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ABSTRACT

This proceedings of the annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education includes the following papers: "Affective Representation and Mathematical Problem Solving" (Goldin); "Cognitive Readiness and Process-Oriented Instruction in Secondary School Mathematics" (Harrison); "An Extended Model of Understanding" (Herscovics, Bergeron); "Using Representations to Conceptualize Teachers' Knowledge" (Orton); "Constructivism, Viability and Values" (Pace); "A Schema Model of Number-Fact Representation" (Baroody); "The Development of Number Concept Using Two Curricular Approaches" (Fischer); "A Study of the Mathematics Behavior of Carrie, a Fourth-Grade Student Introduced to Multiplication" (Landis, Maher); "Mental Computation and Number Sense" (Markovits, Sowder); "The Effects of Structural Variables on the Level of Difficulty of Missing Value Proportion Problems" (Conner, Harel, Behr); "Effects of Type of Numeric Ratio on Strategies Used by Preservice and Inservice Elementary Teachers on Proportional Reasoning Word Problems" (Bezuk); "Curriculum Materials and Common Misconceptions Concerning Division" (Graeber, Baker); "Middle School Students' Partitioning Abilities in Common Fraction Situations--An Investigation in Different Contexts" (Hale); "Concrete Representations for Ratio Reasoning" (Kaput, West, Luke, Pattison-Gordon); "Teaching Fractional Terms to Primary Students" (Larson); "Understanding Decimal Multiplication in Grade Six" (Owens); "A Preliminary Report on the Reliability and Factorial Validity of the Rational Number Thinking Test in the Republic of Trinidad and Tobago (Rahim, Kieren); "Representing and Solving Multiplicative Story Problems as Proportion Problems with a Unit Ratio--A Teaching Equipment" (Sellke, Behr, Voelker); "Solving Story Problems Involving Division with Remainders: The Importance of Semantic Processing and Referential Mapping" (Silver); "Two Students; Two Representations for Fractions" (Tierney); "Numerical Approaches to Algebraic Problem Solving in a Computer Environment (Kieran, Garancon, Boileau, Pelletier); "Teaching Equation Solving: A Teaching Experiment with a Concrete-Syntactic Approach" (Rojano); "Diagrams as an Intermediate Step in the Development of Algebraic Representation" (Simon); "Quantitative Concepts as a Foundation for Algebraic Reasoning: Sufficiency, Necessity, and Cognitive Obstacles" (Thompson); "A Comparison

Among Children, Adolescents, and Adult Learners on Selected Area Measurement Concepts" (Beattys, Pace); "The Development of Geometric Conceptualizations in Logo (Clements, Battista); "Thales's Theorem, Cabri and MacDraw" (Gallou-Dumiel); "Partitioning and Geometry: Is There a Connection?" (Schell, Mason); "Variation in Student Geometric Concepts" (Wilson); "Effectiveness of Small Group Problem Solving Within Classroom Settings" (Alston, Maher); "Cognitive Controls: Are They Related to Mathematics Problem Solving Performance?" (Biggerstaff); "Teaching Experiment Examining the Relationship between Operational Level and Generalizing Ability in the Context of Mathematical Problem Solving" (Gnepp, Maher); "Cooperative Mathematical Problem Solving and Metacognition: A Case Study of Three Pairs of College-Aged Women" (Kroll); "Effects of Extraneous Information and Multiple Steps on the Problem-Solving Performance of Middle School Students" (Muth); "Preservice Elementary Teachers' Critical Thinking Skills and Heuristical Problem Solving Ability" (Oprea, Stonewater); "Multiplication and Splitting: Their Role in Understanding Exponential Functions" (Confrey); "A Construction of the Concept of Exponential Functions" (Rizzutti, Confrey); "'I Haven't Done These Since High School': Prospective Teachers' Understandings of Mathematics" (Ball); "Evaluation of the Mathematical Background of Prospective Elementary School Teachers" (Bergeron, Herscovics); Pre-Service Teachers Conceptions of the Relationship Between Functions and Equations" (Even, Lappan, Fitzgerald); "Geometric Understanding and Misconceptions Among Preservice and Inservice Mathematics Teachers" (Mason, Schell); "A Case Study of a 6th Grade Teacher's Implementation and Evaluation of Lessons for Children in the Learning of Decimals" (Purdy, Maher); "The Early Socialization of Mentored and Non-Mentored Teachers: Two Three-Month Case Studies of Beginning Mathematics Teachers' Professional Networks (Underhill, Brown); "Explicit Knowledge of Infinity" (Wheeler, Martin); "Teacher as Learner; Learner as Teacher" (Weinberg, Parker); "The Influence of Symbolic/Graphic Calculators on the Perceptions of Students and Teachers toward Mathematics" (Dick, Shaughnessy): "A Mathematics Learning Style Instrument Sensitive to Gender Differences in Senior High School and College Students" (Fry); "High School Females' Mathematics Attitudes--An Interim Report" (Gwizdala, Steinback); "Changing Preservice Teachers' Conceptions of Mathematics Learning" (Schram, Wilcox); "An Analysis of In-Service Teachers' Mathematical Beliefs: A Cognitive Development Perspective" (Stonewater, Oprea); "First and Second Graders' Ability to Use Schematic Drawings in Solving Twelve Kinds of Addition and Subtraction Word Problems" (Fuson); "A First Look at the Long Term Effects of Conceptually-Based Instruction on Decimals" (Hiebert, Wearne); "Promoting Good Mathematical Thinking through Writing in the Mathematic Classroom" (Morrow, Schifter); "Overcoming Conceptual Obstacles: The Effectiveness of Conflict Discussion in the Mathematics Classroom" (Onslow); "'Ask Three Before You Ask Me': A Teacher's Attempt to Increase the Role of Peer Involvement in Logo Learning" (Zack); "Integrating Research on Learning and Instruction: A Symposium Presented by Affiliates of The National Center for Research in Mathematical Sciences Education" (Fennema, Carpenter, Brown, Peterson, Thompson, Post, Wearne); "Cognition, Metacognition, and the van Hiele Model of Thinking in Geometry" (Fuys, Lehrer); "An Assessment Instrument to Examine Knowledge of Multiplication and Division Concepts and its Implementation with Inservice Teachers' (Harel, Post, Behr); "A Model for the Development of Leadership and the Assessment of Mathematical and Pedagogical Knowledge of Middle School Teachers" (Lacampagne, Post, Harel, Behr); "Rational Number Knowledge of Preservice Elementary Education Teachers (Cramer, Lesh); "National Center for Research in Mathematical Sciences Education: Curriculum/Assessment Working Group" (Romberg, Kaput, Williams, Zarinnia); "The Teacher's Role in

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Reinventing the Meaning of Mathematical Knowing in the Classroom" (Lampert); "The Situated Activities of Learning and Knowing Mathematics" (Greeno). (MKR)

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Proceedings of the Tenth Annual Meeting

PME—NA

North American Chapter
of the International Group
for the Psychology of Mathematics Education

November 2-5, 1988

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Northern Illinois University

DeKalb, Illinois

North American Chapter
of the
International Group
for the
Psychology of Mathematics
Education

PME - NA

Proceedings of the Tenth Annual Meeting

Edited by
Merlyn J. Behr
Carole B. Lacampagne
Margariete Montague Wheeler
DeKalb, Illinois

3 - 5 November 1988

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PREFACE

PME-NA is at a milestone--the meeting at Northern Illinois University is the tenth annual conference of the North American chapter of the International Group for the Psychology of Mathematics Education. PME was founded in 1976 in Karlsruhe, West Germany, at the Third International Congress on Mathematical Education. PME - NA met for the first time in 1979 and has met yearly since then. The first ten North American meetings have been held in the following cities.

1979 - Evanston, Illinois
1980 - Berkeley, California
1981 - Minneapolis, Minnesota
1982 - Athens, Georgia
1983 - Montreal, Canada
1984 - Madison, Wisconsin
1985 - Columbus, Ohio
1986 - East Lansing, Michigan
1987 - Montreal, Canada
1988 - DeKalb, Illinois

The theme of the meeting is the representation of knowledge for learning and teaching. Perspectives include implications of social contexts on knowledge representation by and for individuals and groups of learners. The plenary speakers, Dr. Magdalene Lampert of Michigan State University and Dr. James Greeno of Stanford University, will direct the conference focus toward theoretical and pragmatic perspectives of the representation of knowledge. The appropriateness of this theme reflects the concern in the research community to investigate the construction, organization, and representation of knowledge taking place within the broad context of the real world.

Seventy-one papers were submitted for presentation at the conference. The papers appearing in this volume were selected by a panel of peer referees. They are clustered in categories reflecting various aspects of the representation of knowledge for learning and teaching and the diverse interests of the researchers attending the conference. The categories overlap, and many papers could have been clustered in several categories. Nonetheless the commonality to be found among the papers is the emphasis on the psychological aspects of mathematics education.

The papers included in these Proceedings support the major goals of the PME-International and PME-North America:

1. To promote international contacts and the exchange of scientific information on the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in these areas with the cooperation of psychologists, mathematicians and mathematics teachers.

3. To further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

The organization and coordination of the conference arrangements would not have been possible without the contributions of Carol Beezhold, Conference Secretary. Our collective thanks go to her.

Merlyn J. Behr
Carole B. Lacampagne
Margariete Montague Wheeler
Northern Illinois University

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AFFECTIVE REPRESENTATION AND MATHEMATICAL PROBLEM SOLVING

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An affective system of representation is joined with several cognitive representational systems in a realistic model for problem-solving competence. The affective states described are not global attitudes or traits, but local changing states of feeling that the solver experiences and can utilize during problem solving--to store and provide useful information, facilitate monitoring, and evoke heuristic processes. We discuss two major affective pathways, one favorable and one unfavorable, together with conjectured relationships between affective states and useful or counterproductive heuristic configurations. Implications of the model include local affective goals for problem-solving instruction.

To model mathematical problem-solving competence and its development realistically, and to formulate teaching goals that maximize enjoyment as well as cognitive ability, we propose that an *affective representational system* (RS) together with several types of cognitive RS's (a verbal/syntactic system; imagistic systems; formal notational systems; and a system of planning, monitoring, and executive control) be included in a unified framework (Goldin, 1987).

The Meaning of Local Affect

Some mathematics education research addresses broad, stable attitudinal constructs which are like *personality traits* in the individual--e.g., degree of independence (McLeod, 1985). Less consideration has been given to *local* affect. By this we mean *changing* states of feeling *during* problem solving. Global affect, we conjecture, results from pathways in local affect which become well established--with connections to the entire framework of cognitive RS's.

Expert problem solvers such as research mathematicians often show signs of intense affect--words, gestures, facial expressions. But most problem-solving research has disregarded affect--not denying its existence, but treating it as a nearly irrelevant concomitant of cognition. If information processing theorists can mechanically simulate problem solving without it, perhaps parsimony suggests we omit it from a cognitive model. However our main conjecture is that affect is not inessential, but *critical to the structure of competencies* accounting for success or failure. Paivio (1978, 1983) advocated a "dual code" model of cognition based on "verbal" and "imaginal" systems, with affect evidently in the imaginal system; but Zajonc (1980) argued for affect as an independent parallel code. But cognitive RS's as I see them *model* problem solving competencies. Thus a question such as the "real" nature of cognitive coding is not relevant--it is unimportant whether affect (or imagery, for that matter) really *requires* another code, a wholly different character set, or is simply a useful (and parsimonious!) summary of complex codes that at another level *do* resemble those of other cognitive RS's. Here the affective RS includes: configurations which are states of feeling; ways that these can influence steps in other RS's; and ways of transforming them based on cognitive occurrences.

Rogers (1983), who generally supports a "triple code" model, suggests:

> "The interactions between the Emotional and other codes are ... intriguing ... Four types of interactions will be discussed: Interruption, Amplification, Diversion and Cognitive Short-Cuts. These four are by no means inclusive, but rather serve to indicate the kinds of processes and structures that could be involved when the Emotional code interacts with the rest of the cognitive system." (p. 292)

The four proposed influences of affect are *general*, applying to almost any cognitive processing under way. While there *are* examples which suggest general rules, such as "the classic 'look away' response to a gory scene" (Rogers, 1983, p. 293) as an instance of Diversion, affective states are more likely to influence cognitions in *domain-specific* ways. Our goal is not to find universal rules for that influence, but to identify *affect-based competencies*--ways that affect is or can be utilized by mathematical problem solvers to guide their steps.

Two Major Pathways Involving Affect and Heuristics

Two well-traveled affective pathways are outlined in Fig. 1. They were not derived directly from clinical research, as we should have ethical objections to intentionally evoking negative affect of the sort described. However they synthesize informal teaching observations and student recollections.

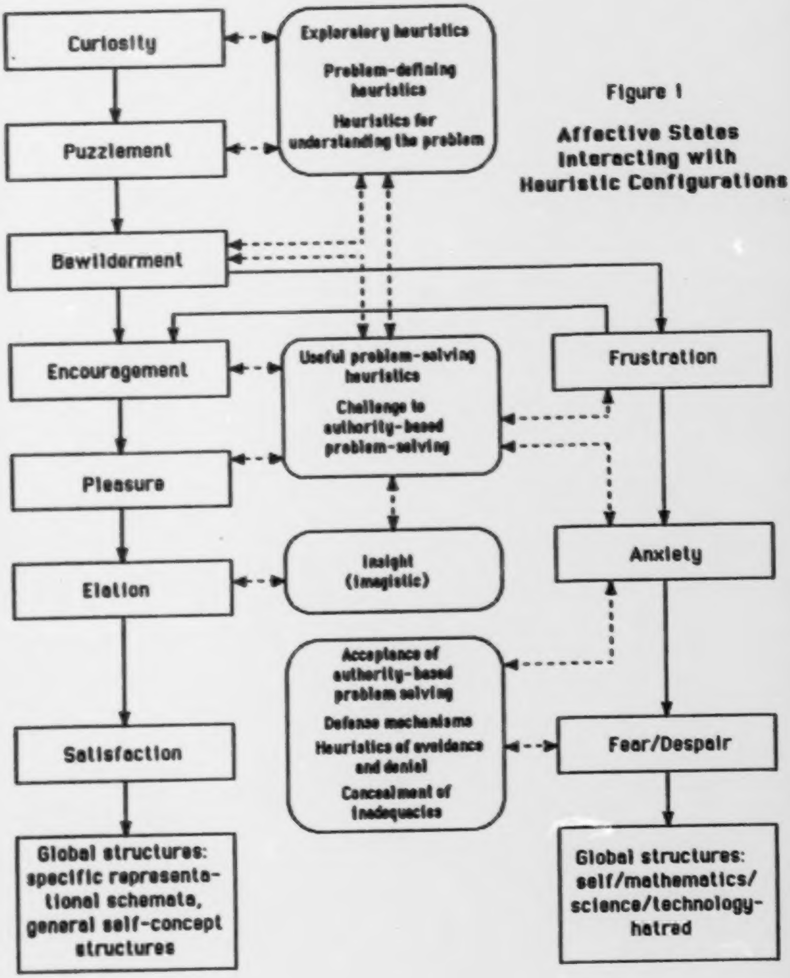
Curiosity
Puzzlement
Bewilderment
Encouragement
Pleasure
Elation
Satisfaction
Global structures: specific representational schemata, general self-concept structures
Exploratory heuristics
Problem-defining heuristics
Heuristics for understanding the problem
Useful problem-solving heuristics
Challenge to authority-based problem-solving
Insight (imagistic)
Acceptance of authority-based problem solving
Defense mechanisms
Heuristics of avoidance and denial
Concealment of inadequacies
Frustration
Anxiety
Fear/Despair
Global structures: self/mathematics/science/technology-hatred

Figure 1

Affective States Interacting with Heuristic Configurations

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In this model initial feelings are of *curiosity*. If the problem has depth for the solver, a sense of *puzzlement* will follow, as it proves impossible to satisfy the curiosity quickly. Puzzlement does not in itself have unpleasant overtones --but *bewilderment*, the next state in the sequence, may: it can include disorientation, a sense of having "lost the thread." If problem solving continues, a lack of perceived progress may result in *frustration*, where the negative affect becomes more powerful and more intrusive; but there is still the possibility that a new approach will move the solver back to the sequence of predominantly positive affect. *Encouragement* can be followed by *pleasure* as the problem yields, by *elation* as insights occur, and by *satisfaction* with a problem well solved. Alternatively, unresolved frustration may lead to *anxiety*, particularly if the context suggests that adverse consequences may result from failure to solve the problem. As the negative affect evolves, we encounter *fear* and *despair*. These sequences of local, transient affective states have consequences --liking or disliking the problem, liking or disliking oneself in relation to the problem solving--that may transfer to other problem-solving situations.

Fig. 1 also shows some conjectured relationships between the affective states and useful or counterproductive heuristic configurations, exemplifying the idea that problem solvers *use* affect to make planning and heuristic choices:

Curiosity. This state can encode the appropriateness of *exploratory* and *problem-defining* heuristics. Divergent inquiry is likely, directed less toward a single stated goal, and more toward finding out what *can* happen in the situation. Heuristic processes such as "Special Cases" may be evoked, for exploratory advance planning reasons. Curiosity stands in contrast with other motivators--practical (the desire to *use* the goal), or authority-based (the need to follow a teacher's instructions or pass a test). Without concomitant curiosity, these might evoke more goal-directed, algorithmic processes. Problem-defining consequences of curiosity may include asking *why* something happens in a situation, seeking the largest domain to which a pattern generalizes, etc. These *create a goal*--a problem where, without curiosity, there would be none.

Puzzlement. This state can encode the fact of unexpected or seemingly contradictory information, or the need to resolve an unanswered question. It can evoke in the competent problem solver a decision to try Special Cases, to Think of a Simpler Problem, to Draw A Diagram, etc.; having as their common element an advance plan of better understanding the problem, of resolving the puzzling features in the special case, the simpler related problem, or the diagram. This outcome would assuage the affect, moving the solver to a state of encouragement.

Bewilderment. This state indicates major unresolved aspects of how to proceed, encoding some failure in understanding, or instability in the solver's current imagistic representation of the problem situation. It can evoke further problem-understanding heuristics, or else "old reliable" methods for solving the problem--e.g., use of an algorithm when a hoped-for shortcut has led to a bewildering conclusion. It can also initiate a healthy *challenge* to *authority*. What the text or teacher has asserted without adequate reason, the bewildered problem solver may question, in effect insisting on constructing a cognitive representation adequate to support or refute the authoritative source.

Frustration. Discouragement and frustration encode lack of progress--e.g., that a process has been retried without success. The affect makes this information immediately available, indeed *intrudes* it rather unpleasantly; but frustration need not be a negative problem-solving outcome. It can still evoke useful heuristics: start over, try a radically new approach or *any* method that will help; try a simpler problem however ridiculous, as long as it is related and solvable; make *any* set of assumptions, no matter how implausible, that would allow a result; etc. There may be no "rational" reason to believe that the process used will work, but "desperate situations require desperate measures."

Anxiety, Fear and Despair. When we move beyond frustration to these states, the now-painful affect may continue to evoke heuristics. But *the problem has been changed*--it is now how to get out of an unbearable situation! The affect is more likely to evoke survival competencies such as avoidance. The least unhealthy response available may now be acceptance of authority-based problem solving, where the solver tries to relieve the anxiety by *complying* --using a rote procedure, guessing a response thought to be desired. If helped at this point, the student's resolve may be to *imitate* the indicated procedure, regardless of "understanding the mathematics." But there can be more severe cognitive consequeces. Powerful *defense mechanisms* can be constructed and evoked in this state of affect, heuristic competencies for avoiding the problem and/or concealing the solver's self-attributed inadequacies. The slow learner delays beginning, fidgets, and needs a glass of water first. The "impulsive" child learns to reach out and grab an arithmetic operation like a lifeline, or to guess at an answer without reasoning; when asked to explain, s/he assumes that the operation or guess must be wrong and offers another one.

Encouragement and Pleasure. The encouragement that follows partial success can evoke continued application of the successful method. It blossoms into pleasure as the method works, heightened when it is the solver's *own* method, not one scripted by authority, that is successful. Evidence for the role of encouragement can be found by observing the consequences of *external* encouragement, where a teacher indicating merely, "you're on the right track," may conquer a student's doubts and enable him or her to persist to a solution.

Elation. This state can encode a breakthrough--the solver's imagistic representation has been drastically reconstructed, with new insights; a formal expression has been greatly simplified through an algebraic technique; or a far-reaching structural relationship has been detected with another problem. Elation may in a sophisticated problem solver evoke processes that search for new interpretations--e.g., to reexamine the problem from the beginning, with the help of the cognitive configurations that led to the feeling of elation.

Global Structures. During problem solving structures are built that the person is likely to retain. A schema applying to a class of related problems has been constructed or brought to bear, incorporating interrelated verbal/syntactic, imagistic, formal notational, heuristic, and now *affective* competencies. In addition, the activity has contributed to the construction of general self-concept structures--a belief in oneself as a good (or bad) problem solver; the expectation of success (or failure) when a mathematics problem is presented, and the anticipation that early feelings of puzzlement, bewilderment or frustration will ultimately result in pleasure, elation, and satisfaction (or anxiety, fear and despair). Thus local affect ultimately leaves its global trace.

Implications for Teaching and Research

The important affective variables go beyond students' attitudes or traits: powerful problem solvers display and use powerful affect. We need research into ways of observing and coding such states and their use. We also need teaching strategies in mathematics in which affect is more than an incidental accompaniment that we try to make pleasant through "motivation."

The model suggests *local affective goals* for problem-solving instruction--e.g., generating problems from the curiosity of students, developing their sense that intense feelings are *appropriate*, and teaching them heuristics to try when these feelings occur. Anxiety, fear and despair, but *not* puzzlement, bewilderment, and frustration, are essentially undesirable affective states. We need to provide productive experiences with and uses of the latter. Students should occasionally experience frustration --not too much, of course--and then be *brought back*, making progress in the problem with processes *suggested* by the negative affect, feeling pleasure, elation, and satisfaction. We give them too little experience with these intensely positive affective states.

Minimal heuristic suggestions at times of discouragement or frustration can be enough to help students solve mathematics problems, transforming the quality of their affect. But if there is already a history of unfavorable affect as in Fig. 1, anxiety and fear evoking avoidance/denial processes can occur almost as soon as a problem is posed. The negative affect and cognitive feedback through avoidance heuristics form a stable structure precluding successful problem solving. The challenge to the educator is to *interrupt the incessant negative feelings*, a first and necessary step in the needed cognitive/ affective reconstruction. The challenge to the researcher is to find ways to do this. One possibility is to change the nature of the goal, from a directed question to an open-ended one. Now as long as the student does *something*, there will be a sense of progress, and curiosity can reemerge. When exploratory success has been encoded affectively as encouragement, more directed questions can be posed.

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COGNITIVE READINESS AND PROCESS-ORIENTED INSTRUCTION IN SECONDARY SCHOOL MATHEMATICS

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The cognitive demands typically made of pupils in secondary school mathematics courses frequently overshoot the cognitive levels at which most are able to operate. Theory-generated remedies for this state of affairs have been developed. Attempts to equip pupils with the mathematical process skills necessary for solving realistic, non-routine mathematical problems are even more demanding. The presentation will address what "mathematical process skills" mean and what is involved in the effective teaching, learning, and evaluating of process skills.

Cognitive Demand/Response Mismatches in Junior High

Richard Skemp has dramatically contrasted **instrumental understanding**, using "rules without reasons," with **relational understanding**, "knowing both what to do and why." (Skemp, 1978, pp. 9, 14). An instrumental approach to mathematics learning is inevitable when the learner is unable to establish connections among various learning experiences, which happens when there is a cognitive "mismatch" between learner and the demands of the learning tasks. A relational approach to mathematics learning engages the learner in building a conceptual structure (schema) from which a variety of strategies for solving any given problem can be selected, modified, and assembled (Skemp, 1978, pp. 9, 14). How to encourage relational learning is suggested by Skemp's description of the way in which "schemas" are constructed by being "built" and "tested." (see Skemp, 1979, p. 21). Such encouragement requires knowledge of the cognitive level at which a pupil can operate in a specific context.

The **Calgary Junior High School Mathematics Project (CJHMP)** (Bye, Harrison, & Brindley, 1980) contrasted the levels of curricular cognitive demand with those of pupil cognitive response in the context of Grade 7 and 8 fraction and ratio topics. The criteria that were used in the CJHMP to assess the cognitive levels of the demands and responses can be described briefly as follows: **Concrete Operational Level**-- able to use ratios and proportions involving 1 to 2, 2 to 1, 1 to 3, 3 to 1, and some other cases, provided countable real objects are available for acting out relationships; **Transitional Level**--able to solve proportions in which the elements in the basic ratio are no greater than 4, able to handle elements greater than 4 when pictorial support with countable parts is provided; **Formal Operational Level**--able to solve proportions in which . . . at least one element is greater than 4, second order proportional thinking is required, symbolic representations of proportional relationships have to be interpreted, or a correct explanation using proportional reasoning is expected. A summary of findings from the Grade 8 **Ratios** cognitive assessments follows:

Distributions by Cognitive Level of Pupil Responses and Curricular Demands: Ratios , Grade 8, N=312

Cognitive Level	**Concrete**	**Transitional**	**Formal**
Pupil Response Distribution:	35%	54%	11%
Experimental Classroom Demands:	16%	77%	6%
Regular Classrom Demands:	0%	0%	100%
Teacher-made Tests:	3%	5%	92%
Textbooks:	0%	9%	91%
Curriculum Guide:	0%	20%	80%

The Grade 8 **Ratios** findings were similar to those for the same topic in Grade 7, and for **Fractions** in both grades. Statistically significant mismatches were found between the distributions of Pupil Cognitive Response ratings and each of the Curricular Demand distributions (Curricular Objectives, Textbooks, Teacher-made Tests, and Experimental & Regular Classroom expectations). The probability that differences as large as those observed would be found in data collected from the same populations was less than 1 in 100. For the majority of the pupils, the curricular demands in fraction and ratio topics were at significantly higher cognitive levels than those at which the students were able to respond. Even though the Experimental Classroom cognitive demand distribution was significantly different from the Pupil Cognitive Response distribution, it was certainly closer to it than were the other demand distributions (Bye, Harrison, Brindley, 1980, pp. 13, 14).

A major focus of the CJHMP study was the assessment of the effects of using a concrete, process-oriented teaching method for the topics in fractions and ratios normally covered in Grades 7 and 8. The Experimental approach was based on the "pupil investigation" model developed by the **South Nottinghamshire Project** (Bell, Wigley, and Rooke, 1978, 1979). The effects of the Experimental (concrete, process-oriented) method of teaching fractions and ratios were compared with those of the Regular , textbook-based approach. In all, 480 Grade 7 pupils, 333 Grade 8 pupils, and 16 teachers were involved in the study, with comparable classes assigned to each treatment. The students were pre- and post-tested on fraction and ratio achievement, attitudes, and "general mathematical strategies." The two topics were taught over a period of twelve weeks in the Grade 7 classrooms and eight weeks in the Grade 8 classrooms. Repeated-measures analyses of variance were used to compare student performances resulting from the two treatments. The concrete, process-oriented teaching approach resulted in significantly improved achievement in, and attitudes towards, fractions and ratios for twelve to thirteen-year-olds. This occurred across a wide ability range. The development of general mathematical strategies was also enhanced, while computational facility was maintained (Bye, Harrison, and Brindley, 1980, pp. 25-50).

Cognitive Readiness for High School Mathematics

When Onslow (1976) assessed the cognitive levels at which Canadian academic-stream **16-year-olds** were operating while reponding to **ratio** tasks, the following response-level distribution was found.

Distribution by Cognitive Level of Pupil Responses: Ratios, Age 16, N=57
(Onslow 1976, p. 37)

Cognitive Level	**Concrete**	**Transitional**	**Formal**
Pupil Response Distribution:	0%	8%	92%

It was found in the Calgary Junior High School Mathematics Project that only 7% of the Grade 7 pupils and 11% of those in Grade 8 were operating at the **formal operational** level when responding to **ratio** tasks. That 92% of a group of students who were three to four years older were found to be operating at the **formal** level on the same ratio tasks indicates a dramatic growth in the cognitive levels at which academic-stream students are able to operate, in the context of **ratios** . It appears that treating topics in **ratios** at a fairly formal level would be appropriate for academically oriented 16-year-olds. Could similar assertions be made for other secondary school mathematics topics, such as **algebra**?

The **Concepts in Secondary Mathematics and Science (CSMS)** findings with regard to response levels of 15-year-olds to **algebra** tasks could be summarized as follows:

Distribution by Cognitive Level of Pupil Responses to Algebra Tasks, Age 15

(Hart, 1981, pp. 116, 117)

Cognitive Level	**Concrete**	**Transitional**	**Formal**
Pupil Response Distribution:	58%	31%	9%

Such findings have important and unavoidable implications for the teaching of high school mathematics. Dietmar Kucheman, the author of the **Algebra** chapter in **Children's Understanding of Mathematics, 11-16** (Hart, 1981), has reported that:

> . . . in algebra and in the other topics investigated, the research has found that children frequently tackle mathematics problems with methods that have little or nothing to do with what has been taught. This may be because mathematics teaching is often seen as an initiation into rules and procedures which, though very powerful (and therefore attractive to teachers), are often seen by children as meaningless. It follows that children's methods and their levels of understanding need to be taken far more into account, however difficult this may be in practice.
>
> On the algebra test the majority of 13, 14 and 15 year olds were at [or below the Late Concrete Operational level--53 to 79%] and were not able to cope consistently with items that can properly be called algebra at all, i.e. items where the use of letters as unknown numbers cannot be avoided. In Piagetian terms these children would seem to be at the stage of concrete operations, which means that for most children the teaching should be firmly rooted in this level whether the aim is to consolidate their understanding or to ease the transition to formal operational thought. (Hart, 1981, pp. 118-119)

The Shell Centre for Mathematical Education at the University of Nottingham has been developing teaching materials rooted in concrete experiences and has been pursuing classroom-based studies with secondary school students on the effects of **Diagnostic Teaching.** At the Shell Centre, Onslow (1986) has used interviews and written tests to identify the misconceptions about rates held by fifty-three 11- to 15-year-olds. Based on the findings, teaching strategies for rates were developed and piloted, and the resulting diagnostic teaching approach was used by regular clasroom teachers in seven experimental classes conducted in parallel with two control classes (There were 157 12- to 14-year-olds, altogether.). Significantly greater gains in overcoming deep-rooted misconceptions like 'multiplication makes bigger' and 'divide the larger number by the smaller' were made by the diagnostic classes than by the expository control group. (Onslow, 1986, pp. 210-213).

Process Skills

Alan Bell has described **mathematical process** in terms of: Abstraction & Generalization, Symbolization & Modelling, and Explanation & Proof (Bell, 1976, p. 3.1, 3.5). Emphasizing **process skills** when teaching and learning mathematics in the secondary school (or beyond) essentially means promoting the development of strategies of abstraction

& generalization, symbolization & representation, and explanation & proof. The development of these poweful process skills necessarily requires learning contexts that are cognitively accessible to the intended learner. What is known regarding the kinds of teaching approach that can foster the development of mathematical process skills across the ability range?

Teaching, Learning, and Evaluating Process Skills

In 1978 and 1979, the South Nottinghamshire Project (based in the Shell Centre for Mathematical Education at the University of Nottingham) produced student mathematical-investigation material to cover the entire mathematics syllabus for 12- and 13-year-olds. The materials were particularly designed to develop mathematical process skills (Bell, Costello, & Kuchemann, 1980, pp. 195, 196)

The British **Joint Matriculation Board** and the **Shell Centre for Mathematical Education** have recently embarked on a landmark collaboration to introduce into the national O-level Examinations (for students of about 16 years of age) questions that will test the **strategic skills** "which are essential if pupils are to be able to deploy their technical skills when tackling unfamiliar problems both in and outside mathematics." These strategic skills are very much the process skills of the preceding section. Recognizing the understandable reluctance of teachers to include more problem solving and investigational work in their mathematics classes when these are not directly tested on examinations, higher level questions are gradually being introduced into the British national examinations by means of a series of "Testing Strategic Skills" modules (containing specimen examination questions and marking schemes; process-oriented classroom teaching materials; and support materials such video and microcomputer software resources). (Joint Matriculation Board & Shell Centre for Mathematical Education, 1984)

In the first "Teaching Strategic Skills" module, **Problems with Patterns and Numbers**, one of the specimen examination questions, **Stepping Stones**, refers to a diagram showing 14 stones arranged in a circular ring with one darkened to indicate where someone hopping around the ring on one foot "starts." A description is given of a girl who changes feet every time she has made 3 hops and discovers that, after she has been around the ring three times, she has stopped to change feet on each of the 14 stones. Then, the following questions are posed:

(i) The girl now hops round the ring stopping to change feet every time she has made 4 hops. Explain why in this case she will not stop on each one of the 14 stones no matter how long she continues hopping round the ring.
(ii) The girl stops to change feet every time she has made *n* hops. For which values of *n* will she stop on each of the 14 stones to change feet?
(iii) Find a general rule for the values of *n* when the ring contains more (or less) than 14 stones. (Joint Matriculation Board and Shell Centre for Mathematical Education, 1984, p. 22)

Each of the "Testing Strategic Skills" questions requires approximately 20 minutes of examination time. The **marking schemes** are designed to give credit for:

[(1)] showing an understanding of the problem,
[(2)] organising information systematically,
[(3)] describing and explaining the methods used and the results obtained,
[(4)] formulating a generalisation or rule, in words or algebraically.
(Joint Matriculation Board and Shell Centre for Mathematical Education, 1984, p. 11)

The marking scheme for **Stepping Stones** assigns credit in three of the four areas by describing how marks would be assigned for partial and full answers for the three tasks posed in the question (i.e., (i), (ii), and (iii)). Following the marking scheme for "Stepping Stones" in the **Problems with Patterns and Numbers** module is a series of actual student answers (in their own handwriting), which are discussed and rated in terms of the marking scheme. A second module in the **Testing Strategic Skills** series has also been produced, entitled **The Language of Functions and Graphs**, with the same general design and features as the one that has just been described. (Joint Matriculation Board and Shell Centre for Mathematical Education, 1985).

The promotion of true mathematical problem solving in secondary school mathematics classrooms may be given considerable impetus if process-oriented questions like those developed at the Shell Centre are included in school-leaving exams.

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AN EXTENDED MODEL OF UNDERSTANDING[1]

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In order to describe the understanding of some mathematical concepts, we now suggest a two-tiered model, one tier identifying three different levels of understanding of the preliminary physical concepts, the other tier identifying three distinct constituent parts of the comprehension of mathematical concepts. The three levels of understanding of the preliminary physical concepts include intuitive understanding, the understanding of logico-physical procedures, and logico-physical abstraction. The three constituent parts of the understanding of a mathematical concept consist of an understanding of the logico-mathematical procedures involved, logico-mathematical abstraction, and formalization. The applicability of this new model is illustrated by using it to describe the understanding of the number concept. Number being perceived as a measure of discrete quantity and of rank, these latter notions thus constitute two preliminary physical concepts.

An extensive survey of proposed models of understanding from 1960 up to 1983 can be found in "Models of Understanding" by Herscovics and Bergeron (1983). Since the early models were heavily oriented towards problem solving and proved inadequate to describe the comprehension involved in concept formation (Bergeron & Herscovics, 1981), we suggested a model which identified four levels of understanding, namely, **intuitive understanding**, **initial conceptualization**, **abstraction** and **formalization** (Herscovics & Bergeron, 1981). But our continued attempt to provide an epistemological analysis of the various conceptual schemata taught at the elementary level, together with relevant research results, have led us to constantly clarify our criteria (Herscovics & Bergeron, 1982; 1983; 1984). Recently, our study of the preliminary physical concepts of the number scheme has forced us to extend our model to include this aspect. Thus, the present paper has two main objectives: first, to present the extended model of understanding; and second, to

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illustrate how this new version of our model can be used to describe the understanding of mathematical concepts. We achieve these two aims by using as an example our analysis of the number concept which provides a paradigm for the new model. The resulting model is a two-tiered model, the initial part describing the understanding of preliminary physical concepts and the second part describing the understanding of the mathematical concept itself.

THE UNDERSTANDING OF PRELIMINARY PHYSICAL CONCEPTS

Since we define 'number' teleologically, that is in terms of its initial uses and functions, we can view it as a **measure of discrete quantity** and as a **measure of rank**. Thus, discrete quantity and rank can be considered as preliminary physical concepts or pre-concepts of number.

Applying our existing model to the notion of discrete quantity and to the notion of rank meant we had to find non-numerical criteria which might be interpreted as representing intuitive understanding, procedural understanding, and logico-physical abstraction of these two concepts. We would not attempt to find a fourth level of understanding, that of formalization, since, in effect, the construction of the number concept could be viewed as the mathematization of discrete quantity and rank. We have been successful in identifying the needed criteria and in converting these into tasks which have been used to assess the kindergartners' understanding of discrete quantity and rank (Bergeron & Herscovics, 1988; Herscovics & Bergeron, 1988).

Intuitive understanding refers to a global perception of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations.

Procedural understanding refers to the acquisition of logico-physical procedures which the learners can relate to their intuitive knowledge and use appropriately.

Logico-physical abstraction refers to the construction of logico-physical invariants (as in the case of the various conservations of discrete quantity and rank), or the reversibility and composition of logico-physical transformations (e.g. taking away is viewed as the inverse of adding to; a sequence of augmentations can be reduced to fewer steps through composition), or as generalization (e.g. perceiving the commutativity of the physical union of any two sets).

As can be seen from the above outline, it is quite possible to identify criteria that will clearly describe three levels of understanding of preliminary physical concepts. These three levels replace advantageously the level of understanding which in our previous model we described as "intuitive understanding of a mathematical concept" since they enable us to provide a full blown epistemological analysis of the preliminary concepts rather than view them as merely the initial embryonic stage in the construction of the intended mathematical concept. Of course, a model of understanding applied to physical notions needs to be distinguished from a model applied to mathematical ones. For instance, the procedural understanding evidenced by the use of a 1:1 correspondence between two sets of objects can be considered as a logico-physical procedure whereas the 1:1 correspondence between objects and the number-word sequence (counting) is of a logico-mathematical nature. A similar distinction applies to the construction of invariants.

THE UNDERSTANDING OF MATHEMATICAL CONCEPTS

We distinguish mathematical concepts from physical concepts when explicit mathematical procedures and invariants are involved. We then can identify three distinct constituent parts of understanding: procedural understanding, logico-mathematical abstraction, and formalization.

Procedural understanding. The procedural understanding of number involves explicit counting procedures. Since we defined number as a measure of quantity and of rank, we had to design various tasks in which all the counting procedures could be

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used. (Fuson (1988) has described different skills in the child's handling of the number-word sequence: reciting from one, counting on from a given number, counting back, etc.). For instance, a task which might favor the counting on procedure was developed: a row of thirteen chips was glued to a cardboard and the first six were hidden in front of the children; they were reminded how many were hidden and then asked how many there were altogether, if they could find the ninth chip, if they could find the rank of an indicated chip.

Mathematical abstraction. In view of our definition of number as a measure of quantity and of rank, the logico-mathematical abstraction of number must reflect both the invariance of quantity and the invariance of its measure, leading to the **abstraction of cardinal number** and it must also reflect both the invariance of rank and the invariance of the measure of rank, leading to the **abstraction of ordinal number**.

Over twenty years ago, Piaget's collaborators Greco and Morf (1962) felt the need to distinguish between 'quantity' and 'the measure of quantity'. They modified the original conservation task involving two equal rows of chips by asking the children to count one of the rows before stretching the other one; they then asked how many chips were in the elongated row while screening it from view. Those who could answer the question were said to **conserve quotity**. Greco and Morf found that many five-year-olds claimed that there were seven chips in each row but that the elongated row had more. Thus, these subjects conserved quotity **without conserving quantity**. For these children, to conserve quotity simply meant that they could anticipate the result of counting the elongated row, but their count was not yet a measure of quantity, since they thought that the quantity had changed. It is only when both quantity and quotity are conserved, when both invariances are perceived, that number becomes a measure of discrete quantity. At that stage, one can claim that the child has achieved a logico-mathematical abstraction of cardinal number. Of course, the Piaget and the Greco and Morf tasks are not the only criteria by which to assess abstraction of cardinal number. These involve but one specific type of transformation. All the other tasks previously used to assess the invariance of discrete quantity could also be used here by modifying them to include enumeration.

An entirely analogous approach can be used to describe the logico-mathematical abstraction of ordinal number. Similar to the notion of quotity, one can introduce its parallel in the context of rank. We define ordity as the ability to predict numerically the rank of an element in an ordered set subject to various transformations such as elongation, translation, hiding part of a row. And of course, there are children who perceive the invariance of ordity without perceiving the invariance of rank. Only when both are present can one claim to have achieved a logico-mathematical abstraction of ordinal number.

Formalization. By the formalization of number, we mean the gradual development of various mathematical notations. When asked to send a message indicating how many objects are in front of them, children will represent each one by a drawing and later on by a tally mark. Once they learn to write their numerals, they may write the sequence 1,2,3,4,5,6,7 to represent the cardinality of a set of seven objects, thereby indicating their need to rely on a 1:1 correspondence between the objects and the numerals; by the end of kindergarten, most of them can use the numeral '7' with its intended cardinal meaning. In fact, many of them can write down numbers exceeding nine. Of course, this does not imply any awareness of place value notation. Nevertheless, it indicates that they perceive the concatenation of two digits globally (e.g. '12' no longer means 'one and two' but 'twelve'). However, even the understanding of positional notation grows gradually: from mere juxtaposition, through a chronological stage, to a final conventional level.

All the tasks we have described have been experimented systematically with various groups of kindergartners and many results have been published. Two papers deal explicitly with numerical procedures (A. Bergeron, N. Herscovics, J.C. Bergeron, 1986; N. Herscovics, J.C. Bergeron, A. Bergeron, 1986a). Results on different tasks dealing with logico-physical abstraction of quantity and logico-mathematical abstraction of number have also been reported (N. Herscovics, J.C. Bergeron, A. Bergeron, 1986b). The kindergartner's symbolization of numbers has been studied and discussed (J.C. Bergeron, N. Herscovics, A. Bergeron, 1986).

DISCUSSION

As can be seen from the above considerations, the understanding of a mathematical concept must rest on the understanding of the preliminary physical concepts. We thus end up with a two-tiered model of understanding. However, this does not imply that the understanding of a mathematical concept needs to await the prior three levels of understanding of the preliminary physical concepts. For instance, our research shows that kindergartners master counting procedures and the formalization of number well before they perceive all the invariances of quantity and rank. It will be noted that we have avoided using the word 'level' to describe the understanding of mathematical concepts and replaced it with the expression 'constituent part' in order to prevent an overly hierarchical interpretation.

We do not claim that this model will be suitable to describe the understanding of all mathematical concepts. Up to now we have applied it successfully to the analysis of the addition of small numbers. Héraud (1988) has applied it to length and the measure of length, to surface and the measure of surface (area). Dionne and Boukhssimi (1988) have applied it to algebraic concepts: to physical point and algebraic point (coordinates); to the physical notion of steepness and to the measure of steepness (slope); to physical straight line and to linear equation.

Our extended model has several pedagogical implications. It links up explicitly the children's mathematics to their physical world and thus strongly suggests using the latter as a starting point in the construction of their mathematical concepts. The new model may also prove to be a better instrument for epistemological analysis. Using such analyses, teachers can develop tasks related to every aspect of understanding of a given concept. They would thus present to the children a far broader range of activities whose complementarity adds up to a much richer cognitive environment.

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USING REPRESENTATIONS TO CONCEPTUALIZE TEACHERS' KNOWLEDGE

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This paper explores Shulman's idea that teachers' knowledge consists, in part, in subject matter representations. If subject matter representations are "knowledge", then they must be justified. Justification is sought, in the case of mathematics teachers' knowledge, by examining representations in mathematics, in mathematics learning, and in cognitive actions. Representations can be justified if mathematics is linked to mathematics learning. However, psychologism must be avoided.

This paper examines the conceptual nature of mathematics teachers' knowledge. A tool that will be used to explore mathematics teachers' knowledge is a subject matter representation. A subject matter representation is a teacher's rendition of a content topic in a way that can be understood by her students (Shulman, 1987; Wilson & Shulman, 1987). Representations are created by asking: "What analogies, metaphors, examples, demonstrations, simulations, and the like can help to build a bridge between [a teacher's] comprehension and that desired for [her] students?" (Shulman, 1987, p. 328). Shulman argues that teachers' knowledge exists, to some degree, as subject matter representations (_ibid_.).

In what follows, "knowledge" will be understood as "justified belief". Thus, if subject matter representations are knowledge, then somehow they must be justified. For example, a teacher might claim to "know" that using fraction bars will help her students learn rational number concepts. If her belief is knowledge, then the belief must be justified. She might argue that fraction bars worked last year, or she might point to research indicating that fraction bars help students learn rational number concepts. Alternatively, she might construct a psychological argument linking the fraction bar representation of rational numbers to student learning.

The guiding question in this paper is: How can mathematics teachers' subject matter representations be justified? The first part of the paper will consider representation in three contexts: in the subject matter (Kaput, 1987a), in mathematics learning (Dienes, 1971; Lesh, 1979), and in cognitive actions (Piaget, 1970). The second part of the paper will examine how these uses of representation in mathematics education might be used to justify mathematics teachers' beliefs.

THREE USES OF REPRESENTATION IN MATHEMATICS EDUCATION

Mathematics can be understood as a discipline whose practitioners represent the real world using concrete or symbolic models (Post, 1981). When a child learns to count, she represents features of the world using number words (numerals). When mathematicians practice their art, they represent groups by other groups, functions by their approximations, and syntactical features by formal models (Kaput, 1987). Logicians reconstruct mathematics using a language or syntax and a model or interpretation of the symbolic expressions. The relationship between formal theory and model can be described as a "representation". Kaput argues that "the idea of representation is continuous with mathematics itself" (Kaput, 1987a, p. 25).

The role that representation plays in the subject matter of mathematics does not necessarily have direct implications for mathematics teachers' knowledge. However, several learning theorists argue that representation plays an important role in the learning of mathematics. Bruner argues that ideas are represented first as actions, then as icons, and finally with symbols (Bruner, 1966). Lesh argues that mathematical ideas can be represented using pictures, spoken symbols, written symbols, actions, and real world situations (Lesh, 1979). Movement or translation among these representations is thought to be an important component of mathematical understanding.

Representation in the learning of mathematics would have implications for teachers' knowledge. A teacher might argue that her selection and ordering of mathematical content enables students to create representations of mathematical ideas and translate between these representations. In other words, she might use ideas of representation in learning theory to justify her subject matter representations.

Justification of teacher beliefs by appeal to learning theory, however, rests on an assumption that needs to be further examined. The assumption is that students learn mathematics by representing mathematical ideas to themselves. This idea is not always accepted in studies of mathematics teaching. Romberg and Carpenter point out that, in studies of teaching, mathematics has been viewed as "a fixed body of knowledge" which must be "absorbed" by the learner (Romberg and Carpenter, 1986, p. 868). This absorption view of mathematics learning can lead to a different conception of teachers' knowledge. Those who ascribe to this view might insist that students of a knowledgeable teacher will be able to "play back" what they have learned.

If the representation view of teachers' knowledge is justified, then it needs further support. Piaget's theory that links mathematics learning and cognitive actions provides some support. Mathematical operations or structures arise from actions which are internalized and then reflected upon (Piaget, 1970). Representation is critical in this process. The learner needs to represent to herself prior actions in order to construct her mathematical knowledge. The knowledgeable teacher would be able to represent the subject matter in a way that enables her students to reflect on their actions and construct mathematical ideas.

The extent to which Piaget's theory of representation might justify teachers' subject matter representations will be examined in the next section.

JUSTIFICATION OF THE REPRESENTATION VIEW OF TEACHER'S KNOWLEDGE

Piaget attempts to tie the representations that are build from reflections on actions to the representations in the subject matter of mathematics. The representations that are used in learning of mathematics (reflections on actions) turn out to be the representations at the foundation of mathematics (the INRC group). In other words, Piaget attempts to explain a structural feature of mathematics using a theory about how mathematics is learned. This attempt to link the structure of a discipline with the learning of the discipline is often called "psychologism", which is a conceptual error. The structural features of a discipline are not necessarily related to the way in which the discipline is created or learned.

When educators attempt to link the learning of mathematics with the subject matter itself, there is always a risk of confounding psychological issues with logical issues. Kaput speaks of the mathematics educator as "always in the mist of Psychologism" (Kaput, 1987b, p. 171). However, psychologism can be avoided by contrasting a mathematician's and a mathematics teacher's understanding of the subject matter. Dewey points to these differences by comparing a scientist's and a teacher's understanding of a discipline:

> For the scientist, the subject matter represents simply a given body of truth to be employed in locating new problems, instituting new researches, and carrying them through to a verified outcome ... [W]hat concerns [the teacher] is the ways in which the subject may become a part of experience; what there is in the child's present that is usable with reference to it; how his own knowledge of the subject matter may assist in interpreting the child's needs and doings ... Thus to see it is to psychologize it... (Dewey, 1902, quoted from Shulman, 1987).

Whereas the concerns of the scientist are the subject matter itself, the teacher is interested in the subject matter "psychologized". Wilson and Shulman support their idea of representation with Dewey's notion of "psychologized subject matter" (Wilson & Shulman, 1987).

Psychologizing a subject matter to teach it is different from psychologizing a subject matter to suggest research problems in the discipline itself. In the former case, the knowledgeable mathematics teacher can be viewed as possessing those subject matter representations that enable her students to reflect on their actions and create their own mathematical knowledge. This can justify the mathematics teachers' representations. In the latter case, however, the structure of mathematics becomes dependent on psychological facts as to how it is learned. This confounds logic with psychology.

SUMMARY AND CONCLUSION

Representation is important in mathematics, in the learning of mathematics, and in cognitive actions. A teacher possesses subject matter representations that enabler her to lead her students to reflect on their actions and construct their own representations. These subject matter representations can be understood as an important component of teachers' knowledge. The representations are justified by linking a theory of cognitive action with a "psychologized" account of the subject matter of mathematics. However, one must avoid claiming that psychologized mathematics exhausts the nature of mathematics itself.

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CONSTRUCTIVISM, VIABILITY AND VALUES

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Constructivist thinkers have tried to describe the complex process of knowledge construction in terms such as "fit" and "viability". While an important discussion has begun, much more needs to be clarified. To help move this dialogue a bit, we introduce the issue of how values might impact on what otherwise may possibly be a purely functional interpretation of the development of knowledge. We suggest that one's values are inextricably bound to the personal development of knowing.

Ernst von Glasersfeld (1981) has described the process of an organism's construction of knowledge in terms of the "viability" of that construction; i.e., as to how interpreted experience is "fit" into an organism's conception of reality, rather than how it is iconically "matched" by that organism with what is described as an unknowable objective reality. This concise formulation is one that is central to a "radical constructivist" perspective and has, for some, helped to clarify one aspect of epistemology. For others this notion seems to offer little but discomfort. Understandably, embracing a conception offering strict limits to the process by which we come to know (Confrey, 1985) seems, at best, a disquieting activity. Nonetheless, there can be solace for some in any struggle to build conceptions-even when a particular conception that we build says that there are implicit limits on all conceptual structures that can be built from our experience.

Unsettling discoverys about our conceptual limits are nothing new. The history of science is one of going bump against limited preconceptions. Copernicus imposed severe limits on man when he dispassionately caste him from the center of Ptolomy's universe. The noneuclideans, like Lobachevsky and Reiman, shattered all sense of the absoluteness of The Geometry. And so too did Godel's incompleteness theorem dash all hopes of conclusively deducing any significant body of mathematics from a finite set of axioms (The Hilbert Program). All such examples as these are cases where understanding virtually leaped forward, at least in part, due to the willingness of some to accept a measure of the emotional discomfort that accompanies a realization that one's previous conceptions are flawed. We note that in each case, man, in his prior knowledge state, was virtually unconscious of the somewhat severe limitations that had always been a part of his conceptions.

Passion, it seems, is a nearly inevitable element of human disagreements. And in each of the celebrated cases of our historical examples, controversy has been notably involved. Thus, that there are currently heated debates regarding constructivism (what it is, if it is, how it is, who it is, and of course, where I can get some cheaply), should be in no way surprising. Consider, for example, that there is a scientific tradition of many centuries during which, from our perspective, "fitters" of knowledge have themselves believed that they have been "matchers" of knowledge. And so it is understandable that such well entrenched practices may popularly continue and, as a consequence, generate substantial resistance to change. Nevertheless, for some, regardless of any attendant difficulties, von Glasersfeld (and others) have asked significant and sobering epistemological questions that, of themselves, are not likely to go away. Perhaps just as importantly, views have been offered which begin to address some of those same questions. As one might suspect, however, the many questions concerning this foundational, but as yet unelaborated, description of epistemological development still far outnumber any supposed answers. Undetered by the difficulties, let us look a bit closer at the term "fit" and the concept of "viability" which we see as being at the core of von Glasersfeld's description of the process of knowledge construction.

THE FUNCTIONAL ASPECT OF FIT

A "fit" is a relation that requires at least two elements to be coordinated. To assess "fit", a cognizing organism must interactionally relate phenomena; i.e. events, to his/her existing conceptions of what such events might plausibly entail. Since any such conception is in some sense, the possession of the knower, there can be no question as to whether or not it is subjective. But what of events? How do we know them? Events occur, and through our observation become what might be called data "fields" (Bornowski, 1956, p.67). Since these events produce data which we can never know in some unadulterated state, completely free of observation and our concepts, they are similarly mediated. Now, since both concepts and events (as data) are thus interpreted, in a logical sense, neither should be expected by the organism to necessarily correctly represent "the truth". They may, or they may not. But, the inquirer cannot know if she/he has found truth through her/his actions. Her/his knowledge must remain tentative. The state of knowing which we describe is not unlike that previously described by the metaphysical realist, Polanyi, as "tacit" knowledge (1958, p.12). But Polanyi, as a "fitter" of knowledge, restricts the label of tacit knowledge to statements about knowledge. In fact, he argues for two kinds of knowledge, an "explicit" or objective kind, and an unformulated or "tacit" kind; the latter of which we discover arising out of our actions. Here we broaden his interpretation. Since it is our major thesis to doubt the ability to know if we have ever obtained Polanyi's objective kind of knowledge, we suggest that the knower must always settle for a kind of validity, likelihood, or perhaps plausibility in this percieved fit between conception and event. Are our conceptions plausible? More importantly, are they increasingly supported by and do they increasingly support data? Do they admit predictability? Perhaps in some process of unerring predictability over time one might argue that an increasingly flawless plausibility, one taken nearly in the sense of a mathematical limit, may allow us to approach certain ontological claims which of late (Kilpatrick, 1987) constructivism has been called upon to provide. That is;

$$\lim_{t \to \infty} (\text{plausibility}) = \text{truth}$$

Thus, it appears that the concept of "fit" is one that can be reduced to a purely functional question. Or can it?

VIABILITY

A thing is normally viable only in relation to the goals for that which it is intended. A chipped coffee mug may make a fine decorative planter. In the absence of a knowledge of purpose, the mug is both viable and not. One should not drink from the mug, nonetheless the plant thrives. It seems that the issue of conceptual viability cannot be resolved in functional terms alone. So too, it follows that "fit", as a concomittant of viability, is similarly open to interpretation. Let's try a few various examples.

In The Past

From the point of view of the slaveowner, slavery was always a system that "worked"; i.e., was viable. There were rulers and ruled and one argument claimed, "that was the way it was supposed to be". So why not, as a slaveholder, own and enjoy the hard labor of others? Further, in the not too distant past, a kind of "research data" was even construed (made to fit) so as to support the absurd notion that the slaves somehow benefitted from that peculiar institution that was slavery (Du Bois, 1964, p.39). However, there can be little doubt that slaves saw in such an unfit societal arrangement, one not viable to their goals.

The point of our example is that viability is mediated by the intervening organism, and in ways that are not always obvious. "...as long as these theories work, the subject will abide by them" (Sinclair, 1987, p. 29). But what "works" for an organism can be made to fit, and thus be defined as viable. Therefore, viability, however much that it might sound purely functional, involves human judgement and as such, is typically rarely value-free. So too, for what is made to "fit" into one's notion of viability. Perhaps through a third example we can clarify a bit further.

In the World

A nuclear bomb is functionally viable if it has a certain explosive force and car destroy accordingly. We dare say, however, that nuclear war is not a viable alternative for humankind. But such a statement is an overt value judgement, rather than one which somehow claims to be devoid of a sense of what is right and wrong. The point is that in all but the most functional cases, values and goals are fundamental in assessing viability. What is "good", "right", "elegant", "of interest", "aesthetic", "natural", etc..., is typically involved in determining our sense of what is viable; that is, what works to move matters toward the ends we seek.

> If we show a man a watch, he will not tend to explain it in terms of machinery, unless he is a watchmaker or a very curious person. He will simply understand it as a timepiece. A watch is defined by its purpose, which is telling time, and that is also the "good" of a watch. Our understanding thus tends to be formulated in terms of the "good". (de Santillana, 1961, p. 213)

In The Classroom

If we narrow the classroom focus and "teach to the test", the children's test scores almost always improve. From one point of view and the values which underlie that perspective, this is a simple and often claimed successful prescription for the improvement of schools. After all, increased test scores give "hard, scientific" evidence that performance has improved. But if we ask the value-laden questions, "Performance toward what goals?", "Education for what purposes?", then the insufficient and transparent nature of such an approach becomes evident. If our job, at best, is to produce well trained test takers or perhaps students with a facility for powerful short-term memory performance, that is one matter. If, on the other hand, like many educators we see our job as one far exceeding the memory training of our charges, then, at least for ourselves, we should seriously grapple with what, in fact, are the educational goals we seek.

In short, can "viability" and its concomittant notion of "fit" escape value judgements? We think not. Consequently, the need, and perhaps, the responsibility to consciously consider underlying goals and values permeates most endeavor, and perhaps doubly so for teaching. Even in a variety of the plethora of scientific research activities which would like to masquarade as somehow "pure", values penetrate unequivocally in the choices which must inevitably be made about what to study and how to study it. In fact, we claim that the very notion that there can exist "pure research" is a value statement about what is good for humans and their science. In addition, the argument that any scientific research can remain somehow removed from human value considerations is one that has arguably not been borne out in practice.

SUMMING UP

So if the viability of a conception too must be interpreted, and as a conception itself, is therefore personal to the knower, how does one determine viablity? While there are no simple answers here, it seems to us that in von Glasersfeld's "fit", or in other words, in the coherence between data and conception, we have an important, if only functional, begining to the discussion. What appears as a tautological relation between concepts that "fit" and those that are "viable", however, can be resolved through an examination of underlying, often implicit values. We see as a crucial element that a learning organism builds conceptions whose "viability" is measured in terms of the value system of that same learner. Our own values thus guide our sense of the viability of our developing conceptions. What "works", works in relation to what we value as represented by our goals. The whole process we describe is one that is thus mediated by the learning organism itself; a self-reflexive process. In addition, we believe that this coherence should demonstrate a functional aspect of its "viability". This means that increasingly more complete sets of data should be subsumed by these conceptions; the growth of experience coherent with preexisting conception- this gives us what are plausible concepts. However, only in a limiting sense can any ontologic claims ever be made.

But perhaps most importantly, we have suggested that "viability" is not purely this functional question. Viability relates function to purpose. A conception is viable in relation to goals or purposes for which it is intended. Goals and purposes, in turn, are a product of the values upon which they rest. Therefore human values and judgement, in the end, will typically determine the viability of a conception. But whose judgement? Who shall decide which values?

Bronowski (1956), has suggested that through the social practice and experience of Science, itself, there have come about certain key values. These include, dissent and the questioning of dogma as an expression of the value of freedom, the honest persuit of "truth", independece of thought and respect for the individual, democracy and respect for the human dignity of others, etc... Science, he believes, gives us a model for values in the last four centuries of its successes.

> Historically, concepts have commonly been set up as absolute and inborn notions, like the space and time which Kant believed to be ready-made in the mind. The view that our concepts are built up from experience and have constantly to be tested and corrected is not classical. The classical view has been that concepts are not accessible to empirica

> tests. How many people understand, even today, that the concepts of science are neither absolute nor everlasting? And beyond the field of science, in society, in personality, above all in ethics, how many people will allow the sanction of experienced fact? The common view remains the classical view, that the concepts of value- justice and honor, dignity and tolerance- have an inwardness which is inaccessible to experience. (p. 50)

It is perhaps in the increased possibility of a more widespread recognition of how values are foundational to our scientific efforts that constructivism has, for some, raised not the pessimistic note previously mentioned, but rather one of optimism. As we have generally recognized the tacit nature of our science, we have helped advance its development. Now, by recognizing what might be implicit value assumptions in our research activity, we make possible a conscious examination of those values. We can thus be empowered (compelled?) to act as only humans seem to be able to; i.e., as humans who can consciously know they are humans (Bronowski, 1973); self-consciously. In the end, we say that this perspective suggests that the measure of human science rests not upon technique, but upon what is unquestionably a human concern; the sense of what is worthy of our human interest. In one sense, in how we define ourselves.

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A Schema Model of Number-Fact Representation

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According to a schema model, children construct schemata that represent important regularities about their experiences with number combinations. Such a model accounts for the development of estimation skill and the mastery of practiced and unpracticed combinations. It also suggests that the mental representation of basic arithmetic knowledge involves both relations (processes) and facts (data).

There is general agreement that practice facilitates mastering the single-digit number combinations (e.g., Nesher, 1986) and that this basic arithmetic knowledge is stored in an associative network (e.g., Ashcraft, 1987; Campbell, 1987; LeFevre et al., 1988; Siegler, 1987). It remains unclear what role *relational learning* and *transfer* play in the mastery process. It is even less clear how the basic number combinations are organized in long-term memory (LTM) and whether or not their representation embodies *relational information.*

While mastering the basic number combinations, children learn fundamental mathematical *relationships* such as the commutativity principle. Currently popular fact-based models (e.g., Ashcraft, 1987) attempt to explain number-fact mastery and representation *without* reference to such relationships (Siegler, 1986). Meaningful-learning proponents (e.g., Brownell, 1935) propose that relational learning is an integral aspect of number-fact mastery. A schema model (Baroody, 1987; Baroody & Ginsburg, 1986) takes this position one step further: relationships may be incorporated into the mental representation of the basic number combinations.

Schema model. A schema is a representation of relationships that summarizes information about many particular cases (cf. Anderson, 1984) and is appropriate for characterizing knowledge that has structure. The essence of number-fact knowledge--like other "consequential knowledge" (Sternberg, 1984)--is structure. A schema model proposes that children construct schemata (prescriptions) to represent the regularities they notice about the number combinations. This implies that they do not make a mental "copy" of each combination but focus on relationships to help them learn and store this information economically (cf. Anderson, 1984).

Mental arithmetic is governed initially by relatively weak schemata--incomplete, inflexible, and disconnected knowledge. Children with little or no understanding of addition may rely on a highly mechanical prescription that yields answers, which to an adult, may make little sense. For example, some children appear to interpret an unfamiliar arithmetic question such as "5 + 2 = ?" as a number-comparison query, which results in the systematic error of responding with the larger addend (e.g., Baroody, in press-b; Ilg & Ames, 1951).

In time, children develop or bring to bear an "incrementing concept": addition involves making a collection larger. Now, at least, they realize that a number needs to be increased. Initially, the prescriptions based upon an incrementing concept may not be very sophisticated. Weaver (1982) points out that because of their incrementing ("unary") view of addition, children may interpret 2 + 4 as "two and four more." As a result, novice adders may simply increase the *first* addend by one (e.g., 2 + 4: "Three"). A somewhat more sophisticated child will realize that a sum must be larger than either addend and use the prescription: increase the *larger* addend by one (e.g., 2 + 4: "Five").

Experience fosters schemata differentiation. For example, at some point, a child may recognize that 1 + 4 and 2 + 4 cannot have the same answer. A new and more accurate prescription is then devised for quickly adding numbers greater than one: add several to the larger addend (e.g., "2 + 4 must be about six or seven"). As children notice other regularities (e.g., addition with zero does not change a number), they construct additional prescriptions. This permits them to respond in an increasingly discriminant and accurate manner (e.g., responding four to 0 + 4, five to 1 + 4, and six, seven, or eight to 2 + 4).

With experience and practice, schemata become more interconnected and automatic and, hence, can provide a basis for more effective problem solving (cf. Anderson, 1984; Greeno, 1978; Kaye, 1986; Norman & Rumelhart, 1975; Resnick & Ford, 1981). More specifically, these principled procedural processes can be compiled and integrated into a production system for efficiently performing complex tasks, such as estimating sums, mastering new combinations, or taking a timed fact test (cf. Anderson, 1983).

Computational estimation. A schema model suggests that before number facts are mastered, mental arithmetic is effectively a problem-solving or *estimating* process. That is, prescriptions such as adding one to the larger addend are a young child's means of using his or her conceptual knowledge to make an "educated guess." In this view, helping children to make more refined educated guesses depends largely on fostering conceptual knowledge and flexibility. Conversely, estimation activities can be instrumental in a meaningful approach to arithmetic learning (Baroody, in press-a).

Transfer of mastery. A schema can provide a basis for assimilating and internalizing previously unknown facts. For example, some children learn a Number-After (N-After) Rule: If an addend is one, then the sum is the number that comes after the other addend in the count sequence (Baroody, 1987). Once children discover this connection between adding one more and the counting sequence, they can use their existing and well learned number-after knowledge to "recall" (new or unpracticed) N + 1 (and later 1 + N) combinations. Noticing that addition is commutative may also promote transfer. For instance, a child might learn 2 + 6 = 8 with little or no practice by recognizing its equivalence with the known fact 6 + 2 = 8. Thus, unlike current fact-based models (e.g., Ashcraft, 1987; Siegler, 1987), a schema model provides the flexibility needed to account for sudden insights and qualitative changes in performance--including transfer to previously unpracticed combinations.

Expert mental arithmetic. Schemata may underlie expert performance on timed fact test by serving as "tests" for evaluating, sorting, and processing different types of problems (cf. Svenson & Hedenborg, 1979). The tests for addition are depicted in Fig. 1. In fact, by screening out nonretrieved responses, Svenson (1985) found that her data formed three plateaus. Combinations involving one and two had relatively quick response times that suggested an incrementating process. These data are consistent with N-After Rule and the Skip-the-Next-Number Rule described in Fig. 1. Moreover, large-addend-first combinations (e.g., 5 + 3 and 9 + 6) had faster latencies than commuted counterparts (e.g., 3 + 5 and 6 + 9) by a *constant* amount. These data suggest that retrieving the sums of small-addend-first combinations involved a process of transforming addend order.

This supports the view that a schema embodying the commutativity principle is an integral aspect in mentally processing basic addition combinations.

Implications. Note that because of a commutativity schema, two related facts may be represented by a single entry in LTM. For example, 2 + 6 = 8 and 6 + 2 = 8 might not be stored as two separate associations but as a single 2, 6, 12 association. Even if some combinations are learned initially as separate facts, acquisition of the commutative relationship may induce a reorganization in the mental representation of basic addition (and multiplication) combinations.

As the addition tests delineated in Fig. 1 become automatic, they can serve as a component for processing other operations. For example, knowledge of addition combinations can serve as data for subtraction (e.g., 8 - 5 is 3 because 5 + 3 = 8). Knowledge of the addition doubles such as 6 + 6 = 12 can serve as the data for the times-two combinations such as 6 x 2 = ?.

Unlike fact-based models that propose a fact-only retrieval system, a schema model proposes that mental arithmetic involves *both* processes and data--both conceptual (relational) knowledge and factual knowledge. Unlike fact-retrieval models that make a sharp distinction between declarative (or factual) knowledge and procedural (or skill-related) knowledge (see Nesher, 1986), a schema model suggests that the distinction becomes blurred. Furthermore, unlike "hybrid models" that suggest a competition (a "horse race") between processes (see Ashcraft, 1985; Kaye et al., 1986), a schema model suggests that factual and principled procedural processes work together cooperatively.

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Figure 1: Addition Schemata[1]

(1) *If* the operation sign is plus, *then* addend order is irrelevant: Focus on the key term regardless of its position (Commutativity Principle).[2]

(2) *If zero* and a number are added, *then* the sum is the number (N + 0 or 0 + N = N Rule).

(3) *If one* and a number are added, *then* the sum is the term after the larger number in the count sequence (N- After Rule).

(4) *If two* and a number are added, *then* the sum is the term after the next term of the larger number in the count sequence (Skip-N Rule).

(5) *If* a number is added to itself, *then* access the associative network (Identity Comparison).[3]

(6) If a *larger number* and a smaller number are added, *then* access the associative network (Magnitude Comparison).[4]

(7) *If* a *larger number* is added to a smaller number, *then* transform the addends and access the associative network (Transformation Process).

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[1] The schemata form a hierarchical checklist by which an expert evaluates addition problems. Schema 1 has priority over the rest.

[2] Unlike subtraction where order is constrained, the commutativity schema permits attention to focus key terms in the following order of importance: zero, one, two, and (large) number.

[3] The associative network would consist of an association-enriched representation of the number sequence.

[4] Making an identity judgment should be easier than determining which term is larger (making a magnitude comparison). Ties often have faster reaction times than nonties. If further work does not confirm a tie effect, Schemata 5 and 6 could be combined: If a larger number and itself or a smaller number are added, then access the associative network.

The Development of Number Concept Using Two Curricular Approaches

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The set-subset relationship is an important mathematical construct that underlies the concept of number, the definition of addition and subtraction, the interpretation of complex addition and subtraction word problems, and the meaning of place value in the base-ten numeration system. The purpose of this study was to compare the effects, in the kindergarten, of a part-part-whole curriculum for teaching the concept of number which stressed set-subset relationships to the effects of the standard curriculum.

A component of number readily recognized by older but not younger children is the perception of a number not only as a whole but also as made up of its parts. Payne and Rathmell (1975) suggested the use of the words whole and parts to emphasize this essential partitioning relationship. A set of objects associated with a number can be separated into subsets, or parts. Each of these subsets may also be represented by a number. For example, seven can be represented as {ooooooo} and also as {ooo/oooo} or {ooooo/oo}. Resnick (1983) stated that a major conceptual achievement in the early school years is the understanding of number as made up of parts.

This Part-Part-Whole (PPW) schema is increasingly recognized by educators as a necessary component of the ability to solve addition and subtraction problems. Using an information processing approach, Riley, Greeno, and Heller (1983) have attempted to model addition and subtraction problem solving. In their model, it is not until the third, and last, level of performance that PPW relationships are established. Therefore, PPW is considered to be a more complex and mature way of interpreting addition and subtraction, freeing the child from the need to rely on direct concrete representations of the specific problem at hand. By enabling the child to see each number or quantity in a problem as a member of a set and also as a member of a subset, PPW allows the child to be more flexible in thinking, using mental counting strategies or derived fact strategies as needed. Further, PPW also allows the child to solve all basic addition and subtraction problems as the differing semantic structures may be interpreted in terms of parts and wholes. (Carpenter, 1986; Cobb, 1987; Romberg & Carpenter, 1986).

Although children need to develop a PPW schema in order to solve the more difficult addition and subtraction problems, instruction in the schema should not be delayed until the addition and subtraction section of the first-grade curriculum. Hiebert and Lefevre (1986) discussed difficulties that arise when children learn information as isolated bits of knowledge. It is important that PPW be a part of the curricular approach to teaching the concept of number in the primary grades so that children recognize the relationships that the number concept has with other mathematical concepts.

Part-part-whole instruction may also reduce the adjustments that children must make when first learning place value. Resnick (1983) recognized the importance of the PPW component of number, when she noted that PPW seemed to be a basic part of the development of an understanding of place value, when children must learn to interpret numbers in terms of a "tens part" and a "ones part". This, followed by the coordination of counting by tens and by ones, is a strong indication of the need for the development of the PPW schema.

The purpose of this study was to compare the effects of a part-part-whole curriculum for teaching the concept of number in the kindergarten to the effects of the standard curriculum. The curricula were assessed by evaluating the children's understanding of the various components of number, their problem-solving performance on addition and subtraction story problems, and their use of the part-part-whole component of number with numbers greater than ten.

For the purposes of this study, number was limited to those components of the concept of number that develop in the young child. The concept was further restricted to number as it relates to discrete sets. Number as a continuous quantity was not included.

The Curricula

Lessons were designed that reflected the two curricular approaches to the number concept. The number of activities, worksheets, and counting experiences provided for the children were controlled across the two curricula, as was the time spent in instruction.

Count/Say/Write. The standard curriculum currently used in the kindergarten for the development of number concepts emphasizes counting, identifying, and writing the numeral that is associated with a set. This standard curriculum was characterized as Count/Say/Write (CSW), since most of the activities in it involve counting discrete sets, saying or recognizing the numbers associated with these sets, and then writing the corresponding numerals. The CSW curriculum used in this study was modeled after this standard curriculum.

Part-Part-Whole. The Part-Part-Whole, PPW, curriculum developed for this study also focused on the cardinal component of number. However, this curriculum emphasized the use of the part-whole component of number to develop cardinal concepts. Instead of children simply counting sets and saying or writing the numerals associated with them, the children explored the various parts into which the sets could be separated. For example, an activity designed to teach the concept of "fourness" would have the children investigate subsets by distributing, in various ways, a set of four objects into two containers. Manipulative materials were used in this exploration, and worksheets were developed to reinforce the concept.

Procedure

A pretest-posttest design was used to assess the effect of the two treatments on a sample of children (n=82) from four kindergarten classes in a predominately white, middle class public school. Two of the classes received the PPW curriculum, and two classes received the CSW curriculum. All the classes were taught by one teacher trained in both curricula. The children received 25 days of instruction. Each daily lesson was approximately 20 minutes long.

The children's conceptual sophistication in terms of various components of number (Number Concept Test), their performance on addition and subtraction story problems (Problem Solving Test), and their ability to use the part-part-whole interpretation of number with numbers greater than ten (Place Value Test) were used as the means of curriculum assessment in individual interviews conducted by a trained examiner.

Results

Collected data were analyzed using multivariate analysis of covariance on the SPSSX statistical package. Results indicated that the group receiving instruction in the PPW curriculum outperformed the group receiving instruction in the CSW curriculum on the Number Concepts Test, the Problem Solving Test, and the Place Value Test ($p < 0.001$). Separate analyses of covariance conducted on the data for each of the tests resulted in significant differences between the treatment groups on each of the tests. Examination of means showed that the group receiving instruction using the PPW curriculum made greater gains than did the groups instructed using the CSW curriculum.

Discussion

The PPW instruction facilitated the development of number concept by kindergarten children. It also enhanced the children's ability to solve addition and subtraction word problems and may provide a conceptual base for emerging informal addition and subtraction strategies. Additionally, early instruction in PPW may expediate instruction in place value and provide a continuity for interpretation of large and small numbers.

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A Study of the Mathematics Behavior of Carrie, A Fourth Grade Student Introduced to Multiplication *

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Observations of the mathematics behavior of Carrie, a nine year old fourth grader, were made over three class lessons designed and implemented by her teacher to introduce different modes of representing the meaning of multiplication. Certain math behaviors were hypothesized as having the potential to occur in the classroom environment created by the teacher. Transcripts of video tapes indicate that Carrie was actively involved constructing solutions to the problems presented, monitoring her own and her partner's work, and utilized several problem solving heuristics.

The theoretical framework for this case study is based on several guiding principles regarding learning and teaching. One is that children can and do learn from each other. Providing opportunities for student interactions in problem-solving tasks make possible the sharing of information and strategies, the monitoring of one's own work as well as that of others, the self correcting of errors, and the supporting of one's own solutions and the questioning of others. They also enable students to be active in their construction of the mathematical ideas, concepts and procedures they are expected to learn. Also, tasks that call for the use of multiple embodiments and/or multiple modes of representation promote opportunities for learners to make connections between and among modes.

These principles are not be considered atomistically but rather as an integrated whole, thus achieving an even greater synergy, making possible the creation of environments that have the potential for building meaningful mathematical learning (Maher, 1988). Activities, intended to provide children with environments for learning mathematics consistent with these principles, were designed and implemented by a fourth-grade

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teacher at the Harding School in Kenilworth, New Jersey (see note 1). The purpose of this investigation was to assess whether or not such learning occurred with Carrie, a nine-year old student. Specifically, the objective was to identify whether certain mathematical behaviors that potentially could occur in this classroom environment, did indeed occur over a three day period.

METHODS AND PROCEDURES

Data Source

For each of three consecutive days, the teacher grouped children in teams for mathematics instruction. Some groups were randomly selected for videotaping. Three researchers independently reviewed the videotaped sessions of Carrie (with her partner) for these lessons in order to identify specific mathematical behaviors, using an evaluation instrument developed for this purpose and the student's worksheets. Segments of tape were excerpted to document occurrence of particular behaviors. Only those segments common to all reviewers provide the data for this report.

The Instrument

An instrument was constructed for purposes of observing and identifying specific mathematical behaviors that might occur in classroom environments that are designed to foster and support children's construction of meaning of the mathematics they are expected to learn. It served as a basis to characterize: (1) the nature of children's interactions with other students and their teacher, (2) the heuristics applied in the solutions of problems, (3) the description of models constructed, (4) the modes of representation called for in the solution of problems, (5) the connections made by children among modes of representation, (6) children's monitoring of their own and classmate's solution, (7) children's supporting and/or challenging their own and other's solutions, and (8) children's making connections between current representations and previous work. When certain behaviors from among the variety of behaviors specified on the instrument were observed, researchers documented their observations with transcribed tape segments.

RESULTS

The Lessons

The three lessons reported in this paper were part of a nine lesson series designed by the teacher to introduce the meaning of multiplication of whole numbers (see note 2). Each day the activity was introduced by presenting the class with a story problem constructed by the teacher to engage the children. Two kinds of concrete materials (plastic chips and cuisenaire rods) were used by the children to build a model of their solution. Three modes of representation (building a concrete model, drawing a picture of the model, and writing a mathematical sentence) were required in children's solutions. Over the three day period, children were grouped to work together with a partner on the activities.

Although space limitations preclude the presentation of all videotaped segments over the three day period that were common to all reviewers as demonstrating the mathematical behaviors of Carrie, their occurrence is reported in Tables 1-3. In Table 1 are the results of observations of the mathematical behaviors of Carrie talking to herself aloud or interacting with her partner and/or teacher. Table 2 indicates the heuristics utilized by Carrie while engaged in solving the problems presented. The representations and/or connections constructed by Carrie while problem solving are given in Table 3.

Table I
Mathematical Behaviors of S1 Working Alone or Interacting with S2 or T

Mathematical Behaviors	Lesson 1	2	3
A) S1 questioned sense of own solution.			X
B) S1 checked accuracy of own solution.	X	X	X
C) S1 questioned sense of S2 solution.	X	X	X
D) S1 checked accuracy of S2 solution.	X	X	X
E) S1 answered questions asked by S2.		X	X
F) S1 explained her solutions to S2.		X	X
G) Sl explained her solutions to T.			X

Note. Sl = Carrie; S2 = Carrie's partner; T = Teacher.

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Table 2
Mathematical Behaviors Utilized by S1 While Problem Solving

Mathematical Behaviors	Lesson	1	2	3
A) S1 drew a diagram.		X	X	
B) S1 thought of a simpler problem.				X
C) S1 thought of a similar problem.				X
D) S1 looked for a pattern.			X	
E) S1 used a guess and check strategy.			X	

Note. SI = Carrie

Table 3
Representations and/or Connections Constructed by SI
While Problem Solving

Mathematical Behaviors	Lesson	1	2	3
A) S1 constructed concrete model.		X	X	X
B) S1 constructed pictorial representation.		X	X	
C) SI wrote a symbolic statement.		X	X	X
D) S1 constructed connections between different modes of representation.		X	X	X
E) S1 made connections between current representations and previous work.				X

Note. SI = Carrie

HIGHLIGHTS OF THE LESSONS

Excerpts from each day's activities are given to demonstrate representative examples of the mathematical behaviors reported.

Lesson 1. The first lesson was introduced by the teacher with a story problem selected from children's literature. The teacher told about Mrs. PiggleWiggle, who had a large bowl of candy that was shared with children who came to visit her in her upside down house. She then simulated the story with a few class members, who were given actual candies after their pretend visit (see note 3). Class members were then asked to pretend that plastic chips provided by the teacher were Mrs. PiggleWiggle's different colored candy. After class discussion, it was agreed that since the candies were of different sizes, it would be fair to receive two red candies if red were chosen; three green candies for green; and four yellow candies for yellow. Presented with similar problems, the children were asked to (1) model the problem using their chips, (2) make a drawing of what they had modeled, and (3) write a number sentence that showed what they had modeled. Carrie working with Rebecca read the problem, "Three children - all took yellow." Although Carrie set up a correct model for this problem using her chips, her picture incorrectly showed 4 groups of four as did her number sentence. Noticing that her answer was different than Rebecca's, she said:

> **Carrie: That's not right . . . . That's not 12.**
> **Rebecca: 4, 8, 12**
> **Carrie: (after rereading the original problem,**
> **she corrected her work)**
> **3 children, all took yellow - OH!**

This video excerpt and the accompanying worksheets provide support that students were indeed involved in the construction of concrete, pictorial and symbolic representations of problems. They also indicate that within this team environment, Carrie monitored her own work as well as the work of her partner.

Lesson 2. On the second day of instruction, the teacher began the lesson by having the children imagine they were going to help Tom build a "jump" for his dog. Using their cuisenaire rods to represent lumber, they constructed dog jumps by placing one rod on top of the next. Later in the activity, students were asked to cover a given rectangle using the longest rods (of the same color) that were possible. Once having done this with the concrete materials, students were asked to draw their rectangle on graph paper. Making a guess that the brown rod would work, Carrie said to her partner:

Carrie: Try brown . . . brown is too long.
Steven: No, try black, black.
Carrie: Black? (After trying it)
O.K. Black.
Steven: Three Blacks.
Carrie:(after using the cuisenaire rods to help her draw lines one by one to fill in the rectangle)
1 ... 2 ... 3 blacks - you're right!

For this activity, children working in pairs used cuisenaire rods to help them solve problems. In addition to the concrete representations that the children constructed, pictorial and symbolic representations were also constructed for the problems. Carrie used a guess and check strategy to help her solve a problem, as well as monitoring her own work and that of her classmate.

Lesson 3. During the third day of instruction, the children worked with the same partners they had the day before, building trains by placing cuisenaire rods one next to the other, and then writing multiplication sentences to find the lengths of the trains. As Carrie and her partner constructed a train of four dark green rods and proceeded to find the length of that train, she remembered they already had constructed a train of two dark green rods, and said:

Carrie: Just a minute, wait. . .
If there was three. . .
Steven: Yeah, there's six in each one, so. . .
Carrie: 6, 12, 18, 24 ... 24 ... so 6 times
Steven: Equals 24
Carrie: 6 X 4 = 24

For this activity, students used cuisenaire rods and worked in pairs to solve problems. Both concrete and symbolic representations for the problems were constructed. Carrie was able to refer to a previous problem as being connected to the one she was currently solving. She was able to explain her strategy to her partner.

CONCLUSIONS AND IMPLICATIONS

The analyses indicated the occurrence of the following mathematical behaviors for Carrie during one, two, or all three of the lessons being reviewed: (1) she thought of a simpler or similar problem to help find a solution; (2) she questioned the reasonableness of her answers and monitored her own work as well as the work of her partner; (3) she used two kinds of concrete materials and drew and labeled diagrams to help find solutions to problems; (4) she made connections between concrete, pictorial, and symbolic modes of representation; and (5) while working with another student, she answered questions asked by her classmate.

While the limitations of forming conclusions based on a single case study are obvious, the analysis of Carrie's mathematical behaviors is an important first step in documenting that these behaviors do indeed occur in natural class settings consistent with the principles described earlier. More work is clearly needed to observe other children with the same teacher and other children with other teachers in a variety of environments in order to gain further insight into the kinds of mathematical behaviors possible.

AUTHOR NOTES

Note 1. The teacher was a participant in the Rutgers/Kenilworth Teacher Development Project in Mathematics (Maher, 1987), a three year program that had as its goal the implementing of the guiding principles described above. Teachers in the project were encouraged to move away from the role of deliverer of information to one of facilitator of children's learning by providing opportunities for children to interact and work in small groups, sharing information and strategies and monitoring their own work as well as the work of their classmates.

Note 2. The three days of lessons might more accurately be considered as an integrated unit designed to introduce children to the meaning of multiplication.

Note 3. The teacher's sensitivity to equity was reflected in her providing each child with some candy at the end of math class enroute to recess.

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MENTAL COMPUTATION AND NUMBER SENSE

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Abstract

Ten fourth-grade and ten sixth-grade students were asked to solve mental computation problems and to tell what procedures they used while working on the problems. The students were interviewed before and after frequent instruction on mental computation over approximately three months. At both grade levels, the percentage of choices to use standard methods to solve problems decreased from before instruction to after instruction, while the percentage of choices to use nonstandard methods increased during the same time period. Standard methods were those based on pencil-and-paper algorithms and focused on digits and rules, while nonstandard methods demanded an understanding of the numbers being processed and usually used a left-to-right procedure and/or a reformulation of the numbers in the problem. It is our contention that the change from standard to nonstandard methods indicates an increased number sense.

Mental computation is once again playing a significant role in the mathematics curriculum, or at least it would appear so from considering its advocates (e. g., National Council of Teachers of Mathematics, 1987) and from looking at scope and sequence charts in newer mathematics texts. The rationale for this renewed emphasis seems to be that many computations are easily done mentally, and that children should be able to choose appropriate means of calculation, whether it be written, mental, or via a calculator, depending upon the situation and complexity of the problem.

An aspect of mental computation that is overlooked in this rationale is its potential for increasing children's number sense (e. g., Plunkett, 1979). In observations of children skilled and unskilled in mental computation, the unskilled are greatly influenced by paper-and-pencil algorithms, and attempt to mentally visualize an entire algorithmic procedure, whereas children skilled at mental calculations make extensive use of place value concepts, spontaneous grouping strategies, and number properties (Ginsburg, Posner & Russell, 1981; Hope, 1985).

There has been little research on the effects of instruction on mental computation, particularly when instruction emphasizes number and operation concepts and treats mental computation as a higher-order thinking skill rather than a rule-based skill. Can children let go of taught algorithms? Will they switch from operating on digits (as in standard algorithms) to operating on numbers (requiring nonstandard procedures)? Can they learn to be flexible in selecting (or inventing) algorithms or strategies appropriate to the problem at hand? If so, how much and what type of instruction is necessary to develop such flexibility?

The purpose of our research is to explore these questions. We are interested not only in tracing the development of skill in mental computation, but also to measure increased understanding of place value concepts, increased flexibility in selecting appropriate computational procedures, and increased use of number and operational properties when carrying out mental computation. The study described here was designed to investigate the effects of daily instruction (for approximately three months) on mental computation.

Procedure

Two fourth-grade and two-sixth grade teachers of average ability classes agreed to participate in the study by providing daily instruction on mental computation. (They were also giving supplemental instruction on number size, another aspect of the research project that included the study described here.) The experimenters provided the background and rationale for increased time on mental computation during a one-day workshop held for the four teachers, and the teachers were provided with detailed lesson plans for several weeks of instruction on mental computation. The lessons were sequential. In both the lessons and teacher workshop it was emphasized that a mental computation problem can be correctly solved in many different ways. Some ways are easier and/or more efficient than other ways, and for each problem a student should think about which way works best for her or him. No rules were taught, although in some instances, for example when multiplying multiples of ten, children were encouraged to find a pattern that could be used on other like problems. The teachers were asked to spend approximately ten minutes per day on the instruction, and to condense or expand the lessons according to the needs of their students.

Ten students from each class were selected for observation throughout the term of the study. All of these students were interviewed before instruction and after instruction. Interviews were audiotaped. The problems given to the students appear in Tables 1 and 2. The interviews also included problems on place value and number size concepts from other aspects of the larger research project.

Results and Discussion

Only two of the four teachers, one at each grade level, taught the mental computation lessons in a manner that encouraged exploring and discussing different strategies. These two teachers used the prepared lessons approximately every other day rather than every day. The analyses presented here are limited to the ten students in these two teachers' classrooms.

Data analyses focused on change in strategies selected. Correctness of the answers, the time taken to solve the problems, and the difficulty level of the problems were analyzed as they related to strategy selection. Strategies were considered to be standard if they were based on standard written algorithms. Students using these strategies tended not to notice relationships between the numbers and instead focused attention on the digits in a right-to-left fashion. For example, a student using a standard strategy on 73 + 19 explained his procedure for getting his answer of 92 by saying he first added 9 and 3, "carried the one," and then added 1 and 7 and 1. Another student using a transitional method began by using a right-to-left algorithm, saying that 3 and 9 is 12, 7 and 1 is 8, and 80 and 12 is 92. Any method that was a variation of the traditional right-to-left algorithm, but which showed some attention to the numbers being processed

Table 1
Mental Computation Methods Selected by Ten Grade Four Students

Problem	Standard	Transition	Nonstandard: No Reformulation	Nonstandard: With Reformulation
		Preinstructional		
12 + 36	8 (8)[a]	-	2 (2)	-
26 + 49	9 (9)	-	-	1 (1)
8 x 60[b]	7 (7)	2 (2)	-	-
70 x 40[b]	5 (0)	4 (2)	-	-
Column total	29	6	2	1
Percent of 40 problems (10 students x 4 problems)	72.5	15	5	2.5
		Postinstructional		
48 + 25	6 (5)	-	2 (2)	2 (2)
73 + 19	4 (4)	1 (1)	2 (1)	3 (3)
23 + 40 + 7	1 (0)	4 (4)	3 (3)	2 (2)
30 + 800 + 70	-	1 (1)	-	9 (9)
84 - 8	1 (1)	-	-	9 (8)
65 - 8	3 (3)	1 (1)	-	6 (6)
30 x 60[b]	5 (1)	3 (2)	1 (1)	-
38 x 2	8 (7)	-	2 (2)	-
Column total	28	10	10	31
Percent of 80 problems (10 students x 8 problems)	35	12.5	12.5	39

[a] Number of correct answers is shown in parentheses.
[b] One student was unable to answer this problem. For this reason, percentages do not add up to 100.

rather than proceeding entirely by school learned rules was considered transitional. In nonstandard methods students did not use the traditional written algorithms but rather focused attention on the numbers to be processed. Students who worked in a left-to-right fashion were considered to be using a nonstandard method with no reformulation. For example, one student worked the above

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Table 2
Mental Computation Methods Selected by Ten Grade Six Students

Problem	Standard	Transition	Nonstandard: No Reformulation	Nonstandard: With Reformulation
	Preinstructional			
26 + 49	8 (8)[a]	-	2 (0)	-
12 x 25	9 (0)	-	-	1 (1)
33 + 19 + 7	6 (4)	4 (3)	-	-
46 + 26 + 3 - 26	7 (2)	2 (1)	-	1 (1)
Column total	30	6	2	2
Percent of 40 problems (10 students x 4 problems)	75	15	5	5
	Postinstructional			
236 + 49	5 (4)	-	4 (4)	1 (0)
76 - 28	2 (0)	-	3 (1)	5 (4)
12 x 25	6 (1)	-	-	4 (2)
33 + 19 + 7	4 (4)	1 (0)	-	5 (5)
27 + 36 + 23	2 (0)	3 (1)	-	5 (4)
73 - 19 + 28 + 19	6 (1)	-	-	4 (1)
248 + 38 - 37	6 (2)	-	-	4 (4)
4 x 16 x 5	2 (0)	-	-	8 (2)
Column total	33	4	7	36
Percent of 80 problems (10 students x 8 problems)	41	5	9	45

[a] Number of correct answers is shown in parentheses.

problem by saying that 70 and 10 is 80 and 3 + 9 is 12, so 92. Other students reformulated the numbers in the problem to make it easier to solve. In one such case, a student said that 70 and 19 is 89, and 3 more is 92. Some problems were designed to test for reformulation, such as those like 33 + 19 + 7, where students could use a standard method by adding the numbers in order ([33 + 19] + 7), or use a transition method by changing the order of operations while maintaining a right-to-left approach ([19 + 7] + 33), or use nonstandard method by looking for the easiest way to arrange the numbers and changing to a left-to-right approach ([33 + 7] + 19). The nonstandard method in this case is based on seeking compatible numbers, which calls for an awareness to how the numbers are related.

The tables indicate that there was movement from selection of standard methods to selection of nonstandard methods between the preinstruction and postinstruction interviews. For example, the selection of standard methods selected by fourth-grade students decreased from 72.5% to 35%, while selection of nonstandard methods increased from 7.5% to 51.5%. Results were very similar at the sixth-grade level.

There appeared to be some relationship between the methods selected and the correctness of the answers obtained. In the 50 postinstructional answers to sixth-grade problems 6, 8, 9, 10, and 11, 23 answers were obtained by students who chose a nonstandard reformulation of the problem, 18 (78%) of which were correct. In the other 27 problems, all solved by other methods, only nine answers (33%) were correct. Other problems did not lead to such clear results, perhaps because some problems were quite easily solved using a standard method, and other problems were not covered in instruction and so did not lead to the use of nonstandard methods.

The time to completion was also affected by choice of method. The mean time for completing the same five problems using nonstandard reformulating methods was 19 seconds, while the mean time for answering the same problems using standard or transition methods was 42 seconds. This relationship was expected, but it was not consistently found for other problems.

The number and kind of operations involved in a problem affected the choice of methods and the correctness of answers. For example, the fourth-grade teacher covered only a few and the sixth-grade teacher covered none of the multiplication lessons, and in both cases students performed poorly on multiplication problems. Another example of problem difficulty would be the effect of the numbers used in similar subtraction problems, such as 84 - 8 and 65 - 8. The results in Table 1 indicate that students find it easier to subtract 4 then 4 more than to subtract 5 then 3 more. A third example appears on the surface to be more of a surprise. Students did not choose to use a left-to-right or reformulation process on addition of two two-digit numbers. Most students were capable of easily and quickly using the standard algorithm and were reluctant to change.

Conclusion

Before receiving the instructional lessons prepared for this study, students' schemas for mental computation were primarily ones in which the paper-and-pencil algorithms were performed mentally, often accompanied with a visual image of the algorithm. The intent of the instruction was to provide opportunities for students to expand these schemas to include not only other ways of doing mental computation, but also to begin to understand that they themselves have to decide what procedure best fits the problem at hand, and that this decision could vary from problem to problem and from person to person. The ability to make the "best" selection depended on factors such as the students' understanding of place value and of how numbers could be reformulated (e.g., when 19 is changed to 20 - 1 to make 73 + 19 easier to solve). Thus, the exploration of different methods can increase number sense, and good choices of methods is dependent on number sense.

The results presented here are based on work done this past year. Plans for next year include further instruction on mental computation in classrooms where at least half of the students whose work is analyzed here will be placed. The influence of expanded schemas of mental computation on the learning of computational estimation will be a major focus of future work.

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THE EFFECT OF STRUCTURAL VARIABLES ON THE LEVEL OF DIFFICULTY OF MISSING VALUE PROPORTION PROBLEMS

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The purpose of this investigation is to observe the effects of structural variables on the level of difficulty of missing value proportion (MVP) problems. Recently, Harel and Behr (in press) proposed a MVP problem difficulty analysis in terms of structural variables. Based on this analysis, we hypothesized a problem difficulty hierarchy for MVP problems in terms of two of these variables, the location of the missing value and the coordination of measure spaces. Problems which reflect these two variables were given to twenty-five preservice elementary teachers. The results of this study will be reported and the possible implications for instruction will be addressed.

Tournaire and Pulos (1985) listed two categories of variables which have been identified as influencing the difficulty level of proportion problems: student centered (e.g., age and gender) and task centered. The task centered variables are of two types: contextual (e.g., mixture versus rate) and structural. The structural variables include integer ratio, order of the unknown and numerical complexity. Harel and Behr (in press) defined an additional structural variable, the coordination of measure spaces, in their analysis of missing value proportion (MVP) problems. In this paper we will propose a MVP problem difficultly hierarchy, with respect to order and measure spaces coordination.

Order of the Missing Value is a structural variable which refers to the location of the unknown quantity with respect to the three given known quantities in the wording of the problem. It is determined by the order of the two per statements--a per b, the closed per statement, and c per ?, the open per statement--in the problem presentation and by the location of the missing value within the open per statement. This variable has four categories: BL, TL, TR and BR, where B (bottom) and T (top) refer to the order of the per statements and L (left) and R (right) refer to the location of the missing value in the open per statement. In the problem: How many candles cost b cents if c candles cost d cents? the location of the missing value is TL (top left).

Coordination of the measure spaces is a structural variable which refers to the syntactical relationship between the units of measures of the four quantities in a MVP problem. The measure spaces in a MVP problem are coordinated when the order of measure spaces referenced in the two per statements is M1 per M2, M1 per M2; they are uncoordinated when the order of the measure spaces mentioned in the wording of the problem is M1 per M2, M2 per M1. For example, 5 candles cost 10 cents, 20 candles cost how many cents? has coordinated measure spaces; but 5 candles cost 10 cents how many cents does 20 candles cost? has uncoordinated measure spaces.

THE STUDY

Careful consideration was taken to hold the other structural variables constant. These include integer ratio (none, one or two), the number of measure spaces (one or two), the number of dimensions of the measure space (one or two) and the partitionability of the measure space (yes or no). For further explanation of these variables see Harel and Behr (in press). The problems used in this study had no integer ratios, two measure spaces, one dimension in each measure space and partitionable measure spaces. The contextual variables were also held constant. The problems used continuous quantities, were of the rate type, were familiar and involved no use of manipulatives.

Problem Types. Eight different problem types resulted from the manipulation of the two structural variables. Four of the problems had coordinated measure spaces and differed in terms of the order of the missing value. The other four problems had uncoordinated measure spaces and differed in terms of the order of the missing value. Since the only variable manipulated in each set of four problems is the order of the missing value, the order of difficulty for each set is the same. There are two factors in this variable, the order of the per statements and the order of the missing value in the open per statement. In terms of the order of the per statements, it is likely easier to solve problems in which the closed per statement appears first because the solver does not have to change the problem presentation. This means that BR and BL problems will be easier than TR and TL problems. Now in terms of the order of the missing value in the open per statement, it is likely easier to solve problems in which the order of the missing value is on the right than when it is on the left. This is because when the missing value is on the right the equation that is formed is of the type $c * v = x$ (v is the computed value from the closed per statement, and $*$ is either multiplication or division), which is a direct computation; but when the missing value is on the left the equation that is formed is of the type $x * v = c$, which is an indirect computation (Harel & Behr, in press). This suggests that BR and TR problems will be easier than BL and TL problems. By combining these two orderings, the complete order of difficulty for each set of four problem types can be hypothesized as $BR < BL < TR < TL$, where $<$ denotes easier.

In order to hypothesize a difficulty order for all 8 problem types, it is necessary to decide whether the coordination of the measure space or the order of the missing value is more salient. At this time there is no indication which that would be, therefore the order of difficulty will be between pairs of problems only. That is, the comparison will be between the coordinated and the uncoordinated problems which have the same order of the missing value. It is likely that the coordinated problems will be easier than the uncoordinated problems because when solving a coordinated MVP problem, the equation formed by the solver for the closed per statement has the same structure as the equation for the open per statement. This reasoning suggests the following order of difficulty: $BL_c < BL_u$, $BR_c < BR_u$, $TL_c < TL_u$ and $TR_c < TR_u$, where c stands for coordinated measure spaces and u stands for uncoordinated measure spaces.

Subjects. The subjects for this study were twenty-five pre-service elementary teachers who were enrolled in a methods of teaching elementary mathematics course at Northern Illinois University. The subjects were given fourteen problems, eight of which were the MVP problem types described above and six were multiplication, division and addition problems which were included to reduce the likelihood of fortuitous correct answers. They were asked to solve each problem and to show all of their work.

RESULTS AND DISSCUSSION

Due to the small sample size, the results indicate the trends identified in the percentage of students who correctly answered the problems. For the coordinated measure space problem types, the results are as follows: $TR_c < BR_c < BL_c < TL_c$. The results for the uncoordinated measure space problem types are as follows: $TL_u < BL_u < TR_u < BR_u$. When comparing the problems which have the same order of the missing value but differ in terms of the coordination variable, the results are as follows: $TR_c < TR_u$, $TL_u < TL_c$, $BR_c < BR_u$, and $BL_u < BL_c$.

Since the results do not match the proposed order of difficulty, it is necessary to identify the error in the assumptions made. Harel and Behr (in press) stated that when solving MVP problems in which the open per statement appears first, it is necessary to change the problem presentation so that the closed per statement appears first. This process is accomplished through problem transformations and results in a problem representation, the solver's conceptualization of the problem. Since the percentage of correct answers for the problems were relatively high, we can assume that the students were able to change the order of the per statements with little difficulty. Therefore, the order of the per statements may be eliminated when considering problem difficulty in MVP problem presentation. When this is taken into consideration we can combine the results of TR with BR and TL with BL which gives the following results: $R_C < L_U < R_U < L_C$.

This indicates that if the measure spaces are coordinated the problems are easier when the order of the missing value is on the right than when it is on the left, $R_C < L_C$. This agrees with the hypothesis. Though, when the measure spaces are uncoordinated the hyposthesis appears not to be supported by the data because when the missing value was on the left the problems are easier than when it was on the right, $L_U < R_U$. It will be agrued that the hypothesis is supported for the problems which have uncoordinated measure spaces.

When solving MVP problems which have uncoordinated measure spaces, it may be necessary as a first step to use a transformation on the problem in order to change the uncoordinated problem presentation into a coordinated problem representation. There are a couple of ways this could be accomplished. For instance, if the student changes the problem so that the unknown is always on the right then $TL_U \rightarrow TR_{C'}$, $BL_U \rightarrow BR_{C'}$, $TR_U \rightarrow TR_{C'}$ and $BR_U \rightarrow BR_{C'}$, where the $\rightarrow$ indicates a transformation and the prime, ', indicates the problem structure under the transformation, the problem represenation. When these new problem representations are substituted in place of the corresponding problem presentations in the results from the uncoordinated set of four problem types, the following order of difficulty occurs: $TR_{C'} < BR_{C'} < TR_{C'} < BR_{C'}$. This result and the assumption that one problem type has consistent difficulty are contradictory, therefore the assumption that the students change the problem so that the unknown is on the right is probably incorrect. Another possible transformation that could be used when solving an uncoordinated measure space MVP problem would change the order of the quantities in the open per statement. The resulting structure changes are as follows: $TL_U \rightarrow TR_{C'}$, $BL_U \rightarrow BR_{C'}$, $TR_U \rightarrow TL_{C'}$ and $BR_U \rightarrow BL_{C'}$. When these new problem representations are substituted in place of the corresponding problem presentations in the results from the uncoordinated set of four problem types, the following order of difficulty occurs: $TR_{C'} < BR_{C'} < TL_{C'} < BL_{C'}$. When the order of the per statements is eliminated from this hierarchy, the resulting hierarchy is $R_{C'} < L_{C'}$. That is, when the unknown is on the right the problem is easier than when the unknown is on the left, which agrees with the results from the coordinated MVP problems. When combining the results for R_C with $R_{C'}$ and L_C with $L_{C'}$, the results are as follows: $R_C < L_C$. Thus the assumption that students change the order of the quantities in the open per statement may possibly be correct.

The analysis of the results reveals that the most salient variable, with respect to percentage correct on MVP problems, is the order of the missing value within the open per statement. This variable not only influences the percentage correct, but also the type of transformation that the student uses on MVP problems with uncoordinated measure spaces. It is also true that once students are able to solve MVP problems, the order of the per statements and the coordination of the measure spaces has little or no affect on problem difficulty.

CONCLUSIONS

The results indicate that the order of the missing value and the coordination of the measure spaces may effect problem difficulty for pre-service teachers. A similar study indicates that the coordination of the measure spaces do effect problem difficulty for seventh grade students (Bezuk, 1986). It is doubtful that these issues are addressed in the classroom. It seems that making students aware of these slight variations in MVP problems would be an easy task for the teacher and may improve performance on MVP problems. It will also help students to realize that the MVP problems have many representations and therefore give them more flexibility in their solution strategies. Teachers can easily manipulate the problem presentation with respect to the structural variables and teach the students the possible transformations that can be applied to the problem presentation to make the problem more easy to solve. There are eight types of transformations, which form a mathematical group under the composition operation. For more information refer to Harel and Behr (in press).

In general, since the percentage correct for all these MVP problems was not impressively high (the highest being 66%) and the subjects were preservice elementary teachers, this signals a need to educate these preservice teachers about MVP problems, their represenatations, the variables which affect problem difficulty and the various solution strategies which are appropriate. A larger study of this design should be pursued in order to validate the findings of this study. Further research should systematically consider all of the variables which have been described in Harel and Behr (in press).

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Effects of Type of Numeric Ratio on Strategies Used by Preservice and Inservice Elementary Teachers on Proportional Reasoning Word Problems

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ABSTRACT: This study examined the effects of the presence or absence of whole number ratios in proportional reasoning word problems on the solution strategies used by preservice and inservice elementary teachers. The type of numeric ratio affected teachers' use of solution strategies in solving proportional reasoning problems, especially their use of the unit rate and factor of change strategies and the cross multiplication algorithm.

What strategies do preservice and inservice elementary teachers use to solve proportional reasoning word problems? How do the numbers in the problem effect their use of these strategies? These questions must be answered in order to plan optimal instruction for these teachers, and to help these teachers provide optimal instruction for their students. The research reported herein examined the effects of the presence or absence of whole number ratios in proportional reasoning word problems on the solution strategies used by preservice and inservice elementary teachers. This research investigated whether the type of numeric ratios effects teachers' use of solution strategies, and whether teachers' strategy use is related to the type of numeric ratios in the problems.

Researchers have found that the presence or absence of whole number ratios affects adolescents' achievement (Karplus, Pulos, and Stage, 1983a & b; Bezuk, 1986) and use of solution strategies (Bezuk, 1986) on proportional reasoning word problems. For example, students use additive strategies more often in problems with non-whole number unit rates; students will often incorrectly reason that, if 2 pieces of gum cost 7 cents, then 3 pieces must cost 8 cents. But the effects of these variables have not been examined similarly with preservice and inservice elementary teachers.

This investigator has observed seven types of strategies used by students when solving proportional reasoning problems. These are an expansion of the strategy categories used by several researchers (Heller et al., 1985, Karplus et al., 1983b, and Noelting, 1980). Table 1 presents examples of each of these seven strategy types.

Table 1 Examples of Each Strategy Type

Example problem: Ann and Kathy each bought the same kind of gum at the same store. Ann bought 2 pieces of gum for 6 cents. If Kathy bought 8 pieces, how much did she pay?

1. Unit rate

   Each piece costs 3 cents, so 8 pieces will cost 24 cents.

   8 pieces * 3 cents/piece = **24 cents**

2. Factor of change

   Kathy bought 4 times as much gum as Ann, so Kathy should pay 4 times as much.

   4 * 6 cents = **24 cents**

3. Cross multiplication algorithm

$$\frac{2\text{ pieces}}{6\text{ cents}} = \frac{8\text{ pieces}}{x\text{ cents}}$$

   2 pieces * x cents = 6 pieces * 8 cents

   2x = 48

   x = **24 cents**

4. Generate pairs

   2 pieces for 6 cents
   4 pieces for 12 cents
   6 pieces for 18 cents
   8 pieces for 24 cents

5. Equivalent fractions

$$\frac{2}{6} * \frac{4}{4} = \frac{8}{24}$$

6. Equivalence class

$$\frac{2}{6} = \frac{4}{12} = \frac{6}{18} = \frac{8}{24}$$

7. Additive

   Ann paid 4 cents more than the number of pieces she bought, so Kathy must have paid 4 cents more than the number of pieces she bought.

   2 + 4 = 6 cents, so

   8 + 4 = 12 **cents**. (incorrect strategy)

The four types of numeric ratios were integral ratios both between and within rate pairs (referred to as "Both Integer"), integral ratios within rate pairs ("Integers Within"), integral ratios between rate pairs ("Integers Between"), and no integral ratios ("Non-integer"). These categories are those used by Karplus et al. (e.g., 1983b) throughout their work. Examples of these four types of numeric ratios are presented in Table 2

Table 2 Types of Numeric Ratios

Problem introduction: Ann and Kathy each bought the same kind of bubble gum at the same store.

1. Both Integer:
   Ann bought 2 pieces of gum for 6 cents. If Kathy bought 8 pieces of gum, how much did she pay? (24 cents)

2. Integers Within Rate Pairs:
   Ann bought 2 pieces of gum for 6 cents. If Kathy bought 5 pieces of gum, how much did she pay? (15 cents)

3. Integers Between Rate Pairs:
   Ann bought 2 pieces of gum for 3 cents. If Kathy bought 8 pieces of gum, how much did she pay? (12 cents)

4. Non-Integer:
   Ann bought 6 pieces of gum for 10 cents. If Kathy bought 9 pieces of gum, how much did she pay? (15 cents)

It was hypothesized that integral unit rates (Both Integer and Integers Within) would promote use of unit rate strategy, that integral factor ratios (Integers Between) would promote use of the factor of change strategy, and that non-integral unit rates and factor ratios (Non-integer) would promote the use of the cross multiplication algorithm and the additive strategy.

Method

The subjects of this study were 228 preservice (N = 175) and inservice (N = 53) elementary teachers who were enrolled in mathematics education courses at either San Diego State University (N = 115) or the University of Minnesota (N = 113). Each subject took a 24-item written test of proportional reasoning ability, each problem containing one of four types of numeric ratios. The test was given before ratio and proportion were discussed in class. The test was based on tests developed by the NSF-sponsored Rational Number Project (Heller et al., 1985) and the problem setting was a buying context. The reliability of the test was found to be .95 by using a Kuder-Richardson 20 (KR 20) estimate of reliability. Scoring of the tests included the classification of the strategy used to solve the problem. A randomly selected subset of 24 proportional reasoning tests (10%) was independently scored as a check on the reliability of scoring. Scoring disagreement or clerical errors occurred on 1% of the problems, indicating a satisfactory level of agreement. The data were analyzed with frequency distributions.

Results

Teachers' use of solution strategies was examined across each of the four types of numbers. Table 3 presents this data.

Table 3 Percent of Teachers Using Each Strategy Type on Each Numerosity Type

	Type of Numeric Ratio			
Strategy Type	Both Integers	Integers Within	Integers Between	Non-Integer
Unit rate	66%	77%	29%	39%
Factor of change	14	2	37	20
Cross mult algorithm	13	14	21	25
Generate pairs	0	0	2	2
Equivalent fractions	1	1	1	1
Equivalence classes	0	0	0	0
Other correct	3	4	7	8
Additive	0	0	0	2
Other incorrect	3	2	3	3

Note: N = 228

Several of the trends observed in this table are of interest. The unit rate strategy was the most frequently used strategy on Integers Within problems and Both Integer problems, which was anticipated because ite unit rate is integral in these problem types. The factor of change strategy used most often on problems with Integers Between, which was anticipated because the factor ratio is integral in this problem type. The cross multiplication algorithm was used often across all four problem types, its use increasing as the numerical complexity of the problems increased. It was a pleasant surprise, however, that the additive strategy was rarely used. The other three strategy types were rarely used.

The use of solution strategies by preservice and inservice teachers was examined across each of the four types of numbers. Table 4 presents this data. Preservice teachers used the unit rate strategy slightly more often than did inservice teachers, while the inservice teachers used the cross multiplication algorithm almost twice as often as did the preservice teachers. The strategy used most frequently by both groups was the unit rate strategy, except in Integers Between problems, when the factor of change strategy was used most often.

The use of solution strategies by teachers in San Diego and Minnesota was examined across each of the four types of numbers. Because no strong trends were observed, that data is not reported here.

Discussion

These findings indicate that the type of numeric ratio affected teachers' use of solution strategies in solving proportional reasoning problems, especially their use of the unit rate and factor of change strategies and the cross multiplication algorithm. These teachers were able to choose from a variety of strategies based on the numeric characteristics of the problems. The use of the unit rate strategy was positively related to the presence of integral unit rates, while the use of the factor of change strategy and the cross multiplication algorithm was positively related to the absence of integral unit rates. Inservice teachers used the cross multiplication algorithm more often than did preservice teachers, while the reverse was true for the unit rate strategy.

Table 4: Percent of Each Group (Preservice or Inservice Teachers) Using Each Strategy Type on Each Numerosity Type

Strategy Type	*Type of Numeric Ratio*			
	Both Integers	Integers Within	Integers Between	Non-Integer
Unit rate	66%	79%	31%	40%
	64	70	23	33
Factor of change	16	2	36	20
	8	2	38	19
Cross mult algorithm	10	11	18	21
	24	24	32	39
Generate pairs	0	0	1	2
	0	0	3	1
Equivalent fractions	1	1	1	1
	0	0	0	0
Equivalence classes	0	0	0	0
	0	0	0	0
Other correct	3	4	8	10
	3	4	4	5
Additive	0	0	0	2
	0	0	0	2
Other incorrect	4	3	5	4
	1	0	0	1

Note: Line 1: Preservice teachers (N = 174)
Line 2: Inservice teachers (N = 53)

These data are different in several ways from that of a similar study with seventh grade students (Bezuk, 1986). Teachers in the present study used the factor of change strategy and the cross multiplication much more than did the seventh grade students, and the teachers used additive and generate pairs strategies much less than did the seventh graders. This may indicate that these teachers have found the two former strategies to be useful and/or easily retained.

These results indicate that many teachers have more than one strategy in their repertoire, and that they are able to examine problem characteristics and choose an appropriate strategy based on those findings. This implies that teachers may be able to teach students how to use various strategies for solving proportional reasoning problems, and that proportional reasoning instruction does not need to be limited to merely teaching the cross multiplication algorithm due to any imagined weakness in teachers' abilities to use or knowledge of other methods. Teachers' understanding of the strategies they use is another important component of their ability to effectively teach these strategies, and will be reported in another article.

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Curriculum Materials and Common Misconceptions Concerning Division

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Theories concerning the conditions necessary for conceptual change are the basis of a number of researched instructional schemes or lesson designs. Misconceptions about division abound, especially when decimals are involved. This study explored the extent to which treatment of division in three recent mathematics textbook series exploits conceptual change theory. The results suggest that the texts implement few of the implications of the theory and accompanying research.

Many people have misconceptions about division that lead to difficulty in solving word problems. These commonly held misconceptions about division include: "the divisor must be smaller than the dividend, the quotient is always less than the dividend" and " the divisor must be a whole number." Investigations of the justifications and the sources of support for these misconceptions suggest that people frequently overgeneralize rules from the domain of whole numbers to the domain of rational numbers. Fischbein, Deri, Nello, and Marino (1985) have argued that the misconceptions follow from the constraints of the primitive partitive and primitive measurement models of division..

The process of overcoming the misconception (or, as some would prefer, personally constructed knowledge that does not conform to public knowledge) is widely acknowledged to be difficult at best. Fischbein argued that "naive intuitive interpretations" are never eradicated altogether. He suggested that we must learn to control the impact of these misconceptions, and that "it is possible that, in the long run, the new logically based interpretation will generate a new intuitive acceptance -- a secondary intuition" (Fischbein, 1987, p. 192).

There are theories about conceptual change including Flavell's (1977) outline of steps needed to reach equilibration and Fischbein's implications for instruction (1987) drawn from his consideration of the role of intuition in mathematics. Research has been conducted on a number of instructional schemes designed to induce and support conceptual change. Two of these instructional frameworks are the "general structure schemes" of Driver (1987) and "conflict lesson" design of Bell et al. (1986).

This study explored the extent to which recent mathematics textbook presentations about the operation of division reflect any of the theory or research on conceptual change. Specifically, we explored the provisions texts made to help middle school students counter the influence of the three common misconceptions about division.

Method

The authors reviewed three textbook series, Addison-Wesley Mathematics (1987), denoted below as Series A; Mathematics Unlimited (1987), denoted as Series B; and Real Math (1987), denoted as Series C. Since these texts were among those recently approved for use in California, they should reflect the emphasis on understanding that the Mathematics Framework requires (California State Dept., 1985, p. 13). In the texts for grades 3-8, each student page was reviewed. All computational exercises requiring division were counted except those (1) needed to rewrite a fraction as a decimal, (2) involving signed numbers, and (3) resulting from or in a percent. Divisions needed in applications were also counted. All division word problems that could be classified as either partitive or measurement were tallied by type. The annotations and suggestions in the accompanying teacher's guide were also analyzed. The questions that guided the analysis were drawn from the theories and instructional frameworks described above.

1) Are students' prior knowledge and the likely resulting conflict noted? At the point when the textbooks introduce problems that violate the constraints of the primitive models, do the textbooks point out the beliefs (the divisor must be less than the dividend, the quotient must be less than the dividend, the divisor must be a whole number) are limited to a specific domain? Are students or teachers informed of the source of these beliefs and the fact that they are quite reasonable beliefs to develop? Or, are students left to realize and cope with the conflict on their own?

2) Is a conceptual meaning provided for the counter-intuitive cases? Are models, stories, or analogies used? Are these models first developed in the familiar domain of whole numbers and then applied to positive rationals?

3) Is there opportunity to apply and reinforce the new concepts? To what extent do the examples and word problems conform to the constraints of the primitive models? For example, is there opportunity for students to apply the knowledge of division by decimals less than one in word problem settings? Are contrasts between

the domain of whole numbers (where the constraints hold) and the domain of positive rationals summarized?

Results

The divisor must be less than the dividend

The confusion arising when "divisor must be less than the dividend" constraint is violated appears to be confounded by students who extend the commutative property to division and by students who are uncertain about placement of the divisor and dividend in the various notations used for division, e.g., $a \div b$, $b\overline{)a}$, and a/b. And there is evidence (Graeber and Tirosh, 1988) that although fourth and fifth students can recognize as division and given prompts can estimate answers (as less than one) for word problems with divisors greater than the dividend, their ability to symbolize such division is weak. These students have "learned about fractions such as 3/4, but they do not recognize this notation as indicated division.

None of the three celebrated, isolated or attempted to give meaning to the first encounter with this type of problem. In each series the first occurrence of a divisor larger than the dividend is with division of monetary amounts over one dollar by a divisor greater than the number of dollars. Series C also presents such problems in grade 4 using divisors that are powers of ten. While Series A and B appear to rely on prior knowledge about money to justify the placement, Series C taught the rationale for this by citing place value. In all three series, non-monetary examples first appear on a decimal division teaching page with no particular attention to the distinction between problems which have a divisor greater than the dividend or a divisor less than the dividend. For both monetary and non-monetary examples the emphasis is given to the process of placing the decimal point and to writing a zero in the ones place, if the quotient is less than one. There is no mention of the relative size of the divisor and dividend; such emphasis might characterize the type of problem with a quotient less than one.

Since there is virtually no attention to distinction between problems with divisors greater than the dividend and those with divisors less than the dividend in either student texts or teacher's guides, it follows that little is done to aid students' conceptual understanding of these problems. Diagrams are not used. The accompanying notation problems are also given little attention. Both Series B and C, on one page of the seventh grade text, include an exercise pointing out that a ÷ b does not equal b ÷ a. The a/b notation for division was introduced in grade 5 in Series B, grade 6 in Series C, and grade 7 in Series A. Once introduced, it always accounts for less than 10% of division notations in the division chapters. Further, although the texts make some attempts to label the terms in division statements as "divisor, quotient, or dividend" the emphasis is on mastery of those terms not on comparing or contrasting the forms.

During the six years encompassed by grades 3-8, Series A and B include approximately 6,000 division computational items; 8% and 12%, respectively involve divisors greater than the dividend. In series C, 13% of the 3,542 division computational items were of this type. However, dramatically fewer opportunities for application are presented in word problem settings. Such problems accounted 12% or less of the division word problems. In the series with the most opportunity over the six grade levels, 79 of the 793 division problems have divisors greater than the dividend.

The quotient must be less than the dividend

This conception, also known as division makes smaller, holds for divisors greater than one. Since division is normally first taught with whole number divisors, prevention of the difficulty is unlikely.

The existence of the misbelief, division makes smaller, and it's source (experience with whole number division) are made clear in Series B teacher's guide, but Series A and C do not mention the misbelief. Unfortunately the material in the guide for Series B is not shared in the student text. Further, the teacher's guide, having noted that the idea of division as sharing will usually not work for a decimal divisor, offers no alternative. The guide notes that teachers should provide plenty of practice for these kinds of exercises that are found to be "such an obstacle for students of all ages" (Grade 6, p. 153A). All three series emphasize the rule for renaming the divisor and proceeding as with whole numbers. The distinction between divisors less than one and greater than one is not made.

Since the measurement model of division can be helpful in giving meaning to division by a decimal less than one, the extent to which this model is established and used is of interest. All three series establish both the measurement and partitive meanings of division. Over the grades 3 to 8, the partitive model is used somewhat more extensively in word problems than is the measurement model. Further, at the point division by decimals less than one is introduced, little is done to suggest that it is the measurement interpretation of division that can give meaning to such division. In Series A, a measurement word problem is used as an example, but the explanation only appears in the teacher's guide. Thus, although a conceptual bridge for linking the whole number and decimal division exists, it is not exploited. What is emphasized is the algorithmic procedure for "moving the decimal point." Although explanations for this procedure are attempted, we found that sometimes the justifications are not very explicitly linked to the procedure with decimal divisors.

The percent of computational division exercises with decimal divisors less than one is 4% or less in all three of the series. Most striking is the number of word problems with a decimal-less-than-one divisor. There is a total across all three series of 39 such word problems. Therefore, even after the skill of division by a decimal less than one is taught, the overwhelming majority of a student's experience is with word problems where "division makes smaller."

The divisor must be a whole number

Another common misbelief about division is that "the divisor must be a whole number." For older students it is unclear whether this belief comes from the corresponding constraint of the primitive partitive model or from adopting the procedural rule, "change the divisor to a whole number," as a belief about the operation itself. Likely the two operate to reinforce another.

As noted above, teachers are, at best, only told that the partitive interpretation is not helpful where decimal divisors are concerned. This misbelief is not described in any of the texts or guides. Any students who are strongly dominated by the partitive model can make little sense of problems such as 4 ÷ .5 or 4 ÷ 1.25 will find no help from the text in coping with their sense of discomfort. Neither will teachers be sensitized to the pupil's possible conceptual difficulties.

Although the measurement model of division is presented and utilized throughout the series examines, in grades 3 to 8 it is not, with the exception of the one word problem example noted above, pointed out as a specific way to bring meaning to examples such as 3 ÷ 1.5. Although Series A and B occasionally and Series C extensively required students to estimate answers to division with decimal divisors, such skills were primarily used as a way of starting the algorithm, that is confirming the placement of the decimal point in the quotient. The use of rounding and mental arithmetic to confirm the reasonableness of an answer is offered as a reason for

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teaching estimation in the guides but is not promoted by the Series A or B student texts. Series C does call for more work in checking the reasonableness of the answer than do the other two.

A total of 7% or 8% of the division computational exercises have a non-integral decimal divisor. There is little opportunity to encounter word problems with non-whole number divisors. The series have a total of 42, 39, and 36 (respectively) word problems with any decimal divisors (both less than or greater than one), across the grades 6 to 8. These constitute approximately 5% to 7% of all division word problems in the texts for grades 3 to 8. Again, the vast majority of all the experience conforms to the belief that the divisor must be a whole number.

Summary

The common misconceptions concerning division are widely recognized and generally attributed to students' conclusions from instruction that, logically, begins with experiences in the domain of whole numbers. While there are theories and a growing number of research studies in science and mathematics education that suggest instructional strategies for supporting conceptual change, recent students texts and teacher's guides exploit few of the suggested strategies. Indeed, the lack of emphasis on meaning for the counter-intuitive cases, the small proportion of examples and word problems for the counter-intuitive cases, and the emphasis on procedural rules may tend to reinforce some of the misconceptions. For example, the procedure of "moving the decimal point," in calculating $.5\overline{)6}$ leads to a final written form $5\overline{)60}$ in which the quotient is less than the dividend and one divides by a whole number. Incidence of word problems that violate the constraint of the primitive models is so low that it is highly unlikely a teacher will ever diagnose a student with common misconception.

If the results are typical of a treatment mathematics receives in most commercial textbooks, theory on conceptual and the accompanying research (albeit limited) apparently have not yet been embodied in the available curriculum. Textbook authors should be encouraged to put the existing theory into practice.

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MIDDLE SCHOOL STUDENTS' PARTITIONING ABILITIES IN COMMON FRACTION SITUATIONS--AN INVESTIGATION IN DIFFERENT CONTEXTS

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This study surveyed middle school students' abilities to partition to solve rational number problems. To describe these abilities, the presence or absence of 22 partitioning behaviors on rational number tasks varying mathematically and perceptually was determined. In addition, one of five partitioning levels was determined per context per subject. Data were analyzed for the sample, for each grade, and for each sex.

Despite the importance of rational numbers, middle school students experience considerable difficulty learning and apply rational number concepts. Unsuccessful efforts to address this difficulty suggest that the problem has been insufficiently assessed and evaluated. Current assessments of students' understanding of rational numbers reveal little about what cognitive mechanisms contribute toward building fraction knowledge. The studies that have looked at these basic cognitive mechanisms have involved very young, very bright children. Kieren (1984) suggests that the role of partitioning in building rational number knowledge may parallel the role of counting in building whole number knowledge. The focus of this study was to describe the status of partitioning of average ability middle school students as they dealt with common fraction situations.

THEORETICAL FRAMEWORK

The mathematical model for this study was derived from Kieren's (1976, 1980) mathematical analyses of rational number into contexts. Instructional models were derived from perceptual and mathematical variability principles of Dienes' (1984) analysis of the role of instruction in rational number building. Finally, psychological models were derived from Piaget and Inhelder's (1971) operative and figurative aspects of tasks and from Pothier and Sawada's (1983) and Piaget, Inhelder, and Szeminska's (1960) delineation of partitioning development into levels.

METHODOLOGY

Subjects were drawn from a middle-class suburban middle school population of 840 students in grades 6, 7, and 8. Two classes per grade, or 155 subjects, comprised the sample. Instruments included Kieren's Rational Number Thinking Test (RNTT) and a set of five structured interviewed tasks. The RNTT was given during two class periods on consecutive days. Twelve representative subjects were individually interviewed.

Three classes of partitioning behaviors were assessed for each of the 7 contexts. These partitioning behaviors, adapted from Pothier and Sawada (1983), included: (1) domain behaviors--the extent to which students dealt with fractions as quantities; (2) construct behaviors--those indicating what mathematics (i.e., number theory, geometry, etc.) was drawn upon to effect the partition; and (3) algorithm behaviors--those indicating kinds of cuts used and procedures such as algorithmic halving.

Contexts of partitioning tasks were varied mathematically and perceptually. Mathematical variations of RNTT tasks included fractions used as part/whole and ratio relationships, quotients, and multiplicative operators. Perceptual variations of interview tasks included fractions in set/subset, area, and length representations.

RESULTS

Partitioning abilities were investigated by determining the presence or absence of 22 partitioning behaviors on 7 variations of rational number tasks. In addition, one of five partitioning levels was determined per context per subject. Data were summarized in frequency percentage distributions for the sample, for each grade, and for each sex.

Partitioning in Four Mathematical Contexts

From RNTT data, context means for attention to quantity, or domain, in all but the quotient context was less than 50 percent for the overall sample. Attention to quantity was least frequent on operator tasks and most frequent on quotient tasks. Grade level comparisons showed significant increase with age in attention to quantity on ratio tasks. Among mathematical constructs used to partition, number theory was the most frequently chosen, especially in the first three contexts where partitioning effectiveness was weakest. Geometry, the second favorite, was dominant in the quotient context where performance was strongest. Algorithm data revealed the halving cut was the predominant choice in the part/whole context.

Partitioning in Three Perceptual Contexts

Attention to domain (or quantity) on interview tasks was significantly higher than that on RNTT tasks. When sample means between domain behaviors were compared, partitioning equivalent parts was found to be significantly lower than all other domain behaviors. However, conservation was attended to significantly more often on interview tasks than on RNTT tasks. On length tasks, parallel slicing was consistently used.

Partitioning Levels in Four Mathematical and Three Perceptual Contexts

These results addressed the quality of students' partitioning behaviors, with Level 1 being the lowest and Level 5 being the highest. Approximately one-half of the sample displayed Level 1 responses to part/whole, ratio, and operator tasks. Performance at higher levels (Levels 4 and 5) was most frequent on interview tasks of set, area, and length where semi-concrete referents were available.

Unhypothesized Findings

The grand mean on the RNTT was found to be only 49.2 percent. Results were lowest for the operator tasks and highest for quotient tasks.

Nonstatistical Findings

Errors common on part/whole tasks were reluctance to partition and an inflexible one-to-one matching. Also, there was evidence of misunderstanding the division concept. On ratio tasks, common errors were failure to focus on relevant variables and inattention to quantity. Consistent with similar studies, errors on operator tasks revealed additive strategies. Finally, on quotient, set, area, and length tasks, algorithmic errors were the most common.

DISCUSSION

It was observed that as contexts became more abstract, there were lower levels of partitioning, but, when semi-concrete materials were allowed, there were higher levels. Also, attention to quantity tended to be more closely related to task success than other partitioning behaviors. Data on ratio tasks suggested that attaining a mature concept of ratio may depend on maturation, whereas choosing appropriate cuts may depend on instruction. It was also observed that partitioning seemed to plateau during the middle school years for this sample. Further, the relatively low status of partitioning ability among middle school students does not appear to fit the level of abstraction expected by current curricula. These observations suggest need evaluations of curriculum and instruction.

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CONCRETE REPRESENTATIONS FOR RATIO REASONING

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We describe the powerful influence of certain concrete representations on students' ratio reasoning patterns. Selections are taken from a typical 7th grade student's interaction with paper versions of visual support materials to show how they abruptly changed his consistently additive strategy to a correct, more multiplicative strategy. Such changes were seen regularly among the larger pool of students who had used such representations in an enactive computer environment. They raise the issue of how to move students from apparently natural additive approaches to quantification to the more potent, but much more difficult multiplicative approaches.

We report on aspects of a highly detailed description of student knowledge of the conceptual field of multiplicative structures, concentrating on the forms of that understanding as students used concrete representations of discrete ratio and proportion. In particular, we focus on describing the interaction between students' performance and certain external representations that we provided - representations described in Kaput, et al (1987) and Kaput & Pattison-Gordon (1987). This interaction illustrates cognitive foundations of ratio reasoning and its conceptual relations with multiplication and division in the discrete case. Our investigation targeted students with a limited, concrete understanding of ratio and proportion. The highly visible impact of concrete visual support suggests foundations for the development of more abstract and flexible understandings of these notions and also reveals how knowledge structures associated with these concepts are related to one another in this complex domain.

THE SUBJECTS AND THE DATA

The subjects included four learning disabled 4th and 5th graders, and eight 6th and 7th graders who were average or below average in school mathematical performance. Only one student had initial competence in ratio reasoning, most had difficulty with division and most had a very limited ability to do rational number arithmetic as measured by written pre-test.

Our data are based on (1) teaching interventions consisting of ten sessions spread over three weeks in an urban school summer program utilizing computer-based learning environments developed over the past three years by the authors; (2) a two-part pretest, a written test of computational skill and an individually administered test of ratio, rate and proportional reasoning whose items were modifications of widely used assessments, e.g., (Hart, 1984; Karplus, et al, 1983); and (3) an individually administered posttest paralleling the second part of the pretest and another augmented version administered to six of the older students six months later. All individually administered tests were audiotaped, as were selected classroom sessions.

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The students' regular school year teachers were interviewed during the autumn following the intervention regarding curriculum and achievement during the previous and current academic years. This helped contextualize views of the students' longer term mathematical learning and problem solving abilities and attitudes.

Although most teaching sessions began with group instruction utilizing a computer projection screen viewed by all the students present, the bulk of class time was spent in individually tutored and observed followup activity. Instruction began with the most concrete of the environments, those that support multiplication and division based on the grouping and manipulation of a single set of discrete screen icon-objects. The older group progressed to the manipulation of two sets of icon-objects in the concrete solution of standard missing value problems (of the type listed below) and then to numerical solutions using tables of data. Only one of the younger group approached ratio reasoning. Recorded teacher and observer debriefing sessions were held daily.

Thus we accumulated a very rich portfolio of descriptive information regarding students' understanding of multiplicative structures and more generally regarding their styles of approach to mathematical problem solving and the assimilation of new knowledge. This paper examines a slice through a single student, with comparisons to the other students as appropriate. Note that the teaching interventions constitute a form of ascertaining experiment rather than a teaching experiment, which would require at least a full year. Such is planned for the future. Lastly, the behavior reported on here is consistent with that observed among approximately one hundred other students who had experience with the representations described below.

THE POWER OF PARTICULAR REPRESENTATIONS

We claim that students' ratio reasoning can be changed in important ways as they act upon a concrete visual representation. We observed supporting instances in two circumstances: (1) using external aids to cope with the complexity of a problem that otherwise stretched their competence beyond its performance limits; and (2) students using some external aid to confron an incorrect pattern of reasoning, such as an additive proportional reasoning strategy (Hart, 1983; Karplus, et al, 1983; Tourniere, et al, 1985). While most of the former such instances can be explained in terms of working memory overloads, they did reveal conceptual inadequacies in interpreting division remainders. We are mainly interested in the latter type of instance, particularly the impact of representations such as that in Figure 1.

Bob's Collision with His Additive Proportional Reasoning Strategy

In both of the post-intervention interviews, almost every student who failed to solve standard (discrete quantity) missing value problems changed their strategy to a successful one when provided a supportive representational framework for their thinking, as in Figures 1 and 2. In most cases such changes occurred without prompting. We describe one interview from the six month followup in detail because it is rich in illustrative power, because of its representativeness, and because the student himself used the concrete representation to explain the shortcomings of his earlier additive strategy.

Among the six students interviewed six months after the intervention, "Bob" was typical in most respects. His school work, was characterized by his summer and his academic year teachers as "average." He was in 7th grade when interviewed for the six month followup. Although the interview was an hour in duration and covered more than two dozen multiplicative problems, we will confine our attention to two sets of standard missing value proportion problems, which we denote as the CIRCUS and TOY problems, respectively.

CIRCUS (1): The circus needs 3 animal trainers for 8 animals. The circus has 15 trainers. How many animals does it have?

CIRCUS (2) assumed 48 animals and requested the number of trainers.

CIRCUS (3) assumed 352 animals and requested the number of trainers.

TOY (1): A toy store sells 4 matchbox cars for every 9 stuffed animals. How many stuffed animals are sold if the store sells 24 matchbox cars?

TOY (2) assumed 72 stuffed animals and requested the number of matchbox cars.

TOY (3) assumed 156 matchbox cars and requested the number of stuffed animals.

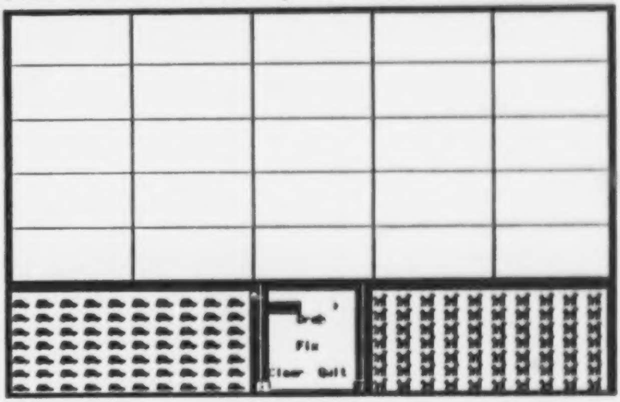

Figure 1

In the absence of visual support, Bob showed consistent and strictly additive behavior on all five problems through TOY (2). Of the remaining five students, one did these problems correctly, one was strictly additive, and three were unsuccessful in other ways. All showed additive behavior on some proportional reasoning problems. Consider Bob on TOY (1):

BOB: [Reading problem] A toy store sells 4 matchbox cars for every 9 stuffed animals. How many stuffed animals are sold if the store sells 24 match box cars? Um, 29.

INT: 29. OK, and how did you get 29?

BOB: I added 5.

INT: You added 5. How did you know to add 5?

BOB: 4+5 is 9.
INT: OK, 4+5 is 9.

After Bob showed analogous behavior on TOY (2), the interviewer brought out a sheet covered with a rectangular grid of empty boxes (the empty version of Figure 2) as part of a protocol that gradually introduced the representations used in the software to determine if and when they cue or otherwise generate more appropriate strategies. Bob, as with most of his peers, did not know why or how to use the empty grid. So then the interviewer introduced the representation in Figure 1, taken directly from software that Bob and the others had used for two sessions six months earlier. He asked Bob to return to TOY (1).

BOB: How many stuffed animals are sold if 24 matchbox cars are sold?
INT: And you also know that a toy store sells 4 matchbox cars for every 9 stuffed animals. How can I use this?
BOB: [Pointing to a left hand column of the empty boxes] Could put 4, 4, 4, 4.
INT: 4, and what would that stand for? What would that 4 stand for?
BOB: The 4 cars.
INT: The 4 cars. OK, you can put 4 in each box, and ...
BOB: Can put them in here?
INT: You can put whatever you want. OK, so you're putting 4 little straight lines into the first three boxes in the first column. How many do you want there? Wait a minute. How many do you want in each box?
BOB: Whoops.
INT: Right, you accidently put three in a couple boxes. OK, so you're crossing out the cars in the warehouse. OK, so you crossed out the 24 cars at the bottom and at the top you have 4 lines in each box. [When using the software version of Figure 1 in two sessions the previous summer, he dragged objects from the lower "reservoirs" into the boxes above.] How did you know how many boxes to fill? How did you know when to stop?
BOB: Because 6X4 is 24.
INT: OK, 6X4 is 24. We know we have 24 matchbox cars. Now what? Now, what do I do?
BOB: Put 9 stuffed animals in each box.
INT: Now I have to put 9 stuffed animals in each box. So now you're drawing circles in the first box. We have 9 in there. You have 9 in the second box. 9 in the third box. 9 in the next one. OK, 9 in the next one. OK, 9 in the next one. OK, good, so now you've filled in all those boxes.

The collision follows. Without writing anything, Bob continues:

BOB: I think I'm off.
INT: You think you're off? Why do you say that?
BOB: Because 9X6 is not 29?
INT: 9X6 isn't 29? Where did you get 9X6?
BOB: 9 in each one, so . . .
INT: There are 9 in each box, uh huh.
BOB: And there are 6 boxes.
INT: And there are 6 boxes.
BOB: 54.
INT: So there are 54. So, which do you think is right?
BOB: 54.
INT: You think 54 is right?
BOB: Yes.

INT: And why do you think the 54 is right?
BOB: Because I did it out. [This response parallels that of other students who regarded their enactive computations as "proof" of their results, needing no further justification. They did not have the same attitude towards the results of a numerically mediated operation.]

Not only did most of the other students change their approach when given the concrete representation, Bob had behaved exactly likewise on his interview six months earlier. This phenomenon is very common and robust in our sample. The interviewer returned to TOY (2).

INT: What about this one here? You know what? I'm going to give you the sheet that has them in it. So, you don't have to draw it out, and see how you have 4 cars and 9 animals. Here is the same thing except it has the 4 cars and 9 animals in each box. OK? So, you don't have to draw it out. [Interviewer gives him the representation in Figure 2.]
INT: OK, now what about [TOY (2)]? How many matchbox cars are sold if 72 stuffed animals are sold?
BOB: Um, [several second pause as Bob reads and then apparently points to a sequence of boxes in Figure 2.] 32.
INT: OK, how did you get 32?
BOB: I did what equals 72, it's 9 times 8. [Missing factor division.]
INT: OK, what equals 72, you said 9X8.
BOB: So it's 8.
INT: OK, so you counted down to the eighth box, and you said 8X4 is 32? How did you know how to do it, that you were supposed to do 8 times 4?
BOB: It's an easy way to count them. [He seemed to use the boxes as indices to support incrementing to the product 8*4.]
...
INT: OK, how many stuffed animals are sold if the store sells 156 matchbox cars? [TOY (3)]
BOB: All of them?
INT: OK, what are you doing?
BOB: I counted all the boxes.
INT: OK, there are 10 boxes [in a column].

He now moved to a higher level of organization, grouping columns of boxes to improve the efficiency of his computation. This process was used by students about one third of the time when larger numbers were encountered.

BOB: 40, 80,
INT: OK, so that's 40 in the first column. 40.
...
BOB: 40, 80, 120.

He eventually counted out 160 cars and then backed up to 156. He then multiplied 39 by 4 to get 156, was asked why he did that, and without answering, went on to compute 39 times 9, getting 351.

BOB: 351. I would've been way off [referring to his earlier answer].
INT: Which way to do you think is right?
BOB: This one [351].

In probing for a fuller explanation of what Bob did, the interviewer inadvertantly provoked Bob to realize what was wrong with his earlier additive strategy; Bob then critiqued that

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strategy decisively - the only student to do so.

BOB: Oh, I know what I did.
INT: What?
BOB: OK, I added 5.
INT: You think you were adding 5?
BOB: Yes, for just like 1 block. [Our emphasis. He was pointing to the boxes.] Another 5, another 5, another 5, another 5, another 5.
INT: It would probably come out right if you took 5 from each box, but you only take 5 once the other way. That's good. So, you can see why adding 5 just once is wrong.

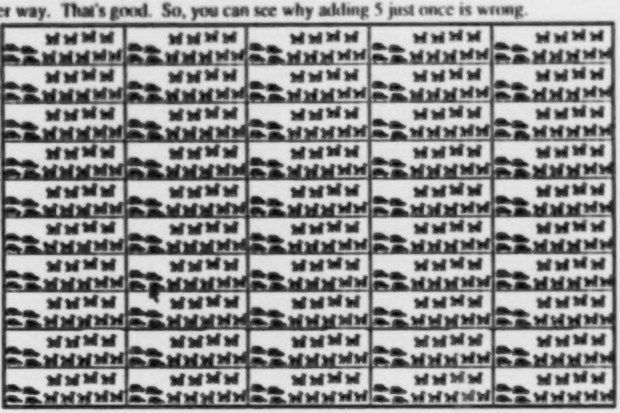

Figure 2

The interviewer then returned him to the Circus problems, which Bob did correctly as well, using a representation analogous to Figure 1 for CIRCUS (1) and Figure 2 for CIRCUS (2&3) and strategies identical to the Toy problems. He had been using a primitive incrementing strategy to determine the number of boxes. The interviewer attempted to determine if Bob was ready to apply his previously exhibited knowledge of division to improve the efficiency of the process. He asked how many boxes are needed for 16 animals and Bob says "2" on the additive basis that "8+8 is 16." Looking for a more multiplicative response, the interviewer eventually asked how many boxes would be needed for 352 animals. Bob then realized with clear surprise that 352 is 8 times 44. The interviewer then asked him how he might determine that if he didn't already know it. Bob needed to be prompted into performing long division to determine the number of boxes.

Disconnected Knowledge Structures

We see two examples of disconnected knowledge structures here. One is a lack of connection between his knowledge of multiplication used to determine the product of the number of boxes with the number of items per box, and the inverse operation of division. Bob used additive incrementing and counting strategies, a common phenomenon in our sample, even when students used missing factor multiplication. Although this primitive version of division is often cited as being tied to a repeated addition model of multiplication (Fischbein, et al, 1985), it was usually used only when prompted. The second missing connection was between his missing factor division and his procedural knowledge of the long division algorithm. We should note that, despite much drill and practice with division algorithms, the students tended to use missing factor division in a modeling situation, i.e., when computing with quantities rather than abstract numbers. Indeed, in two instances students were surprised to learn that they could use long division to solve such problems.

THE REAL ISSUE: MOVING FROM THE ADDITIVE TO THE MULTIPLICATIVE WORLDS

Space limits allow us only to raise, but not discuss, the fundamental questions: *How to extend understanding from the apparently natural additive world to the more difficult to model multiplicative world? And what is the role of the accompanying transition from the discrete to the continuous worlds?*

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TEACHING FRACTIONAL TERMS TO PRIMARY STUDENTS

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In a series of twelve lessons five second-graders and five third-graders were taught part-whole fraction concepts using several area and capacity models. The main purpose of the instruction was for the students to learn and correctly use fractional terms. The students' responses to various tasks presented in pre- and post-interviews comprised the data of this study. Nine of the students improved in associating fractional terms with both area and set models.

The study of fractional numbers (rational numbers when expressed as fractions) is not emphasized in the primary grades. The focus at this level is usually on teaching the students to read and write a few common proper fractions and to associate these fractions with semi-concrete models. It has been suggested by Gunderson and Gunderson (1957) and Larson (1981) that a more natural approach would be for young children to be exposed to many diverse situations that would encourage the development of fraction concepts and an extensive fraction lexicon. The goal advocated is not premature mastery of one or two fraction terms and symbols, but rather children's familiarity with how we can talk about equivalent parts of wholes (units).

This study was designed to describe primary grade students' ability to associate fractional terms and symbols with area and set models for proper fractions, improper fractions and mixed numerals prior to and following instruction in small groups. Since only area models were used in instruction, of special interest was students' performance with set models following instruction.

PROCEDURES

Sample

The subjects consisted of five second grade and five third grade students attending a Tucson school that contained a diverse population at the lower to middle socio-economic levels. Initially ten students at each of the two grade levels were interviewed and then half of the students were selected for instruction. The results of the initial interview with the twenty students was reported at the 1986 PME-NA Conference (Larson, 1986).

Initial and Post Interviews

Two video taped interviews were conducted with each of the ten students one week prior to and two and a half weeks following the instructional sessions. In both interviews students were asked: 1) to construct area and set models when given fractional terms; 2) to produce fractional terms when given area and set models; 3) to read and write fractional symbols, and 4) to produce fractional symbols for area and set models. In the post interview students were also asked to produce the fractional terms for some fraction bars, one of the models used during instruction.

Instruction

The instruction comprised 12 lessons of about thirty-five minutes each over a three week period in March, 1986. The video taped lessons were taught separately to the second and third grade students.

The major emphasis of instruction was to introduce the students to fractional terms for proper and improper fractions and for mixed numerals. A major objective was for students to develop a strategy for determining the name of each equivalent part in a whole (unit) and then to count the parts under discussion in order to produce the appropriate fractional term. Students were also expected to reverse the process in order to select or produce a model for a given fractional term. Fractional symbols were not emphasized in instruction but they were included at various times on bulletin boards, worksheets, and other instructional materials. Usually the written term was included with the symbol.

During the lessons different models and tasks were used to teach the objectives and to encourage generalizations beyond the actual models used. The following concrete materials were used: circular and rectangular regions, measuring cups, pattern blocks, and fraction bars. Some games and worksheets were used to provide practice in associating fractional terms with semi-concrete and concrete models.

RESULTS

Producing Fractional Terms

Nine of the ten students were more familiar with the meaning of fractional terms following instruction. The one third-grade boy who did not perform better on these tasks was absent for four of the twelve instructional sessions. In the initial interview the only fractional terms produced for area and set models were: one-half (area, 70%; set, 30%), one-quarter (area, 20%; set, 10%), and three-quarters (area, 20%). In the post interview all of the students identified one-half of a region and 70-90% correctly produced the following terms when shown area models: two-thirds, one-fourth, two-fifths, and one and one-third.

Students were less successful in producing fractional terms for set models than for area models but they did increase their production. Terms correctly used to describe set models were: one-half or two-fourths (70%), one-third (70%), three-fourths (60%), five-sixths (30%), and two and one-half (30%).

In five of the twelve lessons, fraction bars were used as models in different teaching activities. When asked in the post-interview to tell the part of the bar that was shaded black, only the two third-graders who were absent for part of the instruction were not successful on this task producing only one or two of the six terms tested. All of the other eight students correctly produced five of the six terms tested: *one-fourth*, *three-thirds*, *zero-fourths*, *four-sixths*, and *seven-twelfths*. The three third-graders and two of the second-graders also correctly produced the term *one and two-thirds*.

Constructing Models for Fractional Terms

In the initial interview students were very successful in constructing both set and area models for *one-half* (area, 100%, set 90%). Only two students could construct models for other fractional terms. In the post interview, there was a 90% success rate for showing one-third of a region and two-thirds of a set. Seven of the students could model three-fourths of a region and two students though unsuccessful with this task could show one-fourth. Six of the ten students could show one-fourth of a set of four. Forty percent and 20% respectively could model *two and one-half* and *six-fourths* with regions.

Reading and Writing Fractional Symbols

Most students improved in their ability to read and write fractional symbols. Overall scores for each of these sections of the interview are: 1) write fractional symbols, pre-score, 34%, and post-score, 77%; and 2) read fractional symbols, pre-score, 45%, and post-score, 74%. The order of difficulty for writing the symbols was the same on both interviews. The mixed numeral 1 1/3 (pre 10%, post 30%) was the most difficult and 1/2 (pre 30%, post 60%) was more difficult than all of the other fractions including the improper fraction 7/4 (pre 40%, post 90%). A similar order occurred in the reading task but the differences were not as great.

Writing Fractional Symbols for Models

Initially third-graders were more successful than second-graders in writing fractional symbols for area and set models. There was little change in third-graders performance in the post interview (area 60%, 60%; set 20%, 40%), whereas the second-graders improved from 20% correct responses to 70% with area models and 0% to 60% with set models.

CONCLUSION AND DISCUSSION

In the teaching activities used in the instructional phase of this study fractional terms were emphasized. Fraction symbols were used at various times but were seldom the main focus of instruction. In spite of the instructional emphasis, correct responses to tasks involving reading and writing fractional symbols and associating fractional symbols with models increased, especially for the second-graders. Students' ability to produce fractional terms for area and set models increased drastically. Individual students differed greatly in what they learned and in their consistency across tasks. Except for one third-grade boy who was absent for one-third of the sessions, all of the students improved in using fractional terms to describe area and set models and in producing models for specific fractional terms.

Additional research is needed on the long range effect of systematic instruction of the type used in this study. Perhaps if students were given the opportunity to develop a fraction lexicon and basic fraction concepts over a long period of time without a premature emphasis on mastery and fraction symbols, they would have a foundation at the intermediate grade levels for understanding the more complicated aspects of the rational number system.

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Understanding Decimal Multiplication in Grade Six

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The purpose was to study the acquisition and understanding of decimal multiplication concepts. A small-group teaching experiment on multiplication was conducted with a group of six grade 6 students after the investigator had taught background decimal concepts to the whole class . Results from the written tests and interviews were mixed. Estimation of products was satisfactory in one type, but not in another type of exercise. In choosing the operation for a problem, having the answer will assist students in confirming the choice. The result of an experience with a diagram was applied to a computational question. Students need many experiences in many and varied settings to understand how whole number arithmetic generalizes to the rationals.

Under what conditions would we be willing to say that learners in school or even adults understand "multiplication of decimals"? One thing it could mean is getting an answer to a computational exercise like 2.71 X 0.54. However, this kind of exercise is often done as a rote procedure, and we might say "without understanding." We might be more willing to say that a student understands "multiplication of decimals" if he or she can estimate the product in this exercise. Another ability we could agree to call "understanding" is to be able to know when to use multiplication (Bell, et al., 1984). That is, one understands multiplication if one can apply it in an actual or contrived problem situation. Clearly, these two or three kinds of "understanding" are qualitatively different.

The purpose of this paper is to report an in-depth teaching experiment of children's understanding of multiplication of decimal fractions. Skemp (1976) contrasts instrumental understanding and relational understanding. Resnick (1982) indicates that carrying out the steps of an algorithm (syntactic rules) may or may not reflect an underlying semantics, that is, meaning. Hiebert and LeFevre (1986) have discussed procedural versus conceptual knowledge. While these theorists embody slightly different shades of meaning in these constructs, the contrast remains valid. The study reported here investigates the child's deeper understanding of decimal fractions and multiplication of decimal fractions in particular.

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A split grade 5/6 class containing 14 grade six students was made available for study. In order for students to undertake multiplication of decimals substantive background was necessary. The investigator led the entire 5/6 class in experiences in decimal concepts, notation and equivalence using base 10 models (4 sessions), addition and subtraction introduced with base 10 (7 sessions), order and equivalence using number lines and linear metric measure (3 sessions) and measurement applications (3 sessions).

At this point the project continued with the target group of six grade 6 children. This group was instructed by the investigator in another room, while the rest of the 5\6 class remained with the regular classroom teacher and used the project materials. The target group continued with the following experiences: equivalence of common and decimal fractions (1 session), "of" means multiplication (e.g., 6 m ÷ 2, 1/2 of 6 m, 0.5 of 6 m, 0.5 x 6 m, 4 sessions); rounding and estimating products (3 sessions); drawing area diagrams to find products (2 sessions) and developing rules like "tenths times tenths is hundredths" (1 session); and finally writing expressions to indicate the operation in multiplication or division problems (1 session). Instruction was designed to illustrate several meanings of multiplication of decimals. The usual computational rule about counting decimal places in each factor and in the product was withheld until after the data were collected. It was felt that the children might short-circuit understanding and prefer to latch onto a rule. Following instruction, tests were given and interviews were held with each target student.

Four of the 14 subtests (51 items total) will be discussed in this paper. A sample of **Mult** (6 items) is, Solve: .54 X 2.1 =____. Estimate the answer, then place the decimal point in the given "answer." 7.342 X 0.5 = 3 6 7 1 is an example of **EPDP** (4 items). **Multprob** (10 items) and **Divprob** (4 items) were together in one set of story problems where the students were asked to supply the "equation" (expression giving the operation) or select the correct expression from a list of choices. Sample items will be shown in the discussion of the interviews. **Area** is a subtest of three items. For example, Joe's back yard is 25.4 m long and 4.3 m wide, what is the total area of his back yard? Pretests and posttests were identical except that the area items were not on the pretest.

RESULTS

Table 1 presents the subtest scores for the six target students on both the pretest and the posttest. They are listed on the order of high, middle and low pairs in terms of their original selection for the project.

Table 1

Pretest and Posttest Data with Number of Items for Six Target Students

		Mult(4)	EPDP(4)	MultProb(10)	DivProb(4)	Area(3)
Aaron	pre	2	3	6	3	--
	post	3	4	10	1	3
Candice	pre	0	2	4	0	--
	post	1	4	8	0	2
Frances	pre	0	0	4	0	--
	post	0	2	7	1	2
Shayne	pre	0	0	1	0	--
	post	4	0	4	0	1
Greg	pre	0	0	3	0	--
	post	2	2	4	2	0
Jennifer	pre	1	3	0	0	--
	post	2	2	2	0	0

Placing the decimal point

In addition to data from the written test of four items, two of the same items were included in the interview setting. In the first item, place the decimal point in 3.25 x 5.52 = 2 1 1 9, all six of the target group got it correct. Generally, students find it easy to learn to estimate products when the factors round to small whole numbers. The other item, stated earlier (7.342 x 0.5 = 3 6 7 1) was another story. On the second try, Aaron computed 7 x 0.5 to get it correct. After trying 7 x 0 = 0 and 7 x 1 = 7, Candice tried "fractions" at the interviewer's suggestion. While she placed the point correctly, she was unable to justify it. After trying 7 x 1 = 7, Frances could not capitalize on the suggestion to use fractions. After noticing 7 x 1 = 7, Shayne thought to place the decimal after the 7. When the interviewer questioned it, he recalled, "When you multiply something that's less than zero [i.e., one], you get some number that's smaller," and he got it correct. Greg wrote, "7.000 x 1.0 = 7.0," and he was not successful with the suggestion to try fractions. Jennifer said, "There is one number before the decimal place. So, I think it [the product] would be the same," (that is, 3.671). In summary, this problem was very difficult. Only Aaron, could apply the strategy to round to one significant digit to compute the estimate. Admittedly, the time spent on this extension of estimating products was inadequate.

Choosing the Operation

In addition to written test data on choosing the correct operation (see MultProb and DivProb in Table 1), the interview included five story problems. In the first case the students were asked to solve the problem. In the next two cases they were asked to provide an "equation" (expression) and in the last two, to select the correct expression. In each case the operation was to be identified. The statement of each problem is given in Table 2. The choices for selecting operations are not shown. The numbers indicate their placement on the written test. The last interview problem was not on the written test.

Table 2

Interview Problems for Choosing the Operation

16. On a highway, a four-wheel drive car can go 7.5 km on each litre of gasoline. How many kilometres can the car be expected to go on 1.3 litres?

24. Tom spent $900.00 for 0.75 kg of platinum. What would be the price of a 1 kg bar of platinum?

22. To make a wallet-size picture of Bryan Adams from a poster, it has to be decreased to 0.052 of its original height. If the poster of Bryan was 1.2 metres high, what will the height of the picture be?

35. If 16 friends got together and bought 4 kg of cookies, how much would each of them get?

A teacher is pacing up and down the classroom. Each pace is 0.74 metres in length. How many paces will she make in walking the full length of the classroom, 9.2 metres?

When Aaron was not confident of the logic of the problem, he would form an hypotheses and get a result to test it. In two cases, he suggested "divide," once in the wrong order, but he was unable to complete the computation. His estimation skills were excellent. His idea about the size of the answer consistently influenced his choice of operation. In one case, he drew a diagram and chose easier numbers: How many paces of 40 cm across a 3 m room?

Candice multiplied in the gasoline consumption problem because she expected the answer to be larger. On the platinum problem she wrote, "$900.00 x 0.15 kg." She expected the answer to be about $100 more than $900. She might have been thinking, but she did not verbalize $900 + 0.25 x $900. On the poster-picture, she wrote "0.052 ÷ 1.2 metres." Upon request for alternate notation, she wrote, "$0.052\overline{)1.2m}$." While not verbalized coherently, it seems that she was thinking that division makes smaller. On the cookies she chose $4\overline{)16}$, but she was quite unsure, because the answer would be 4, and "this might be wrong." In the pacing problem, she chose 0.74 x 9.2, because "It will add up to how many paces she will need in the classroom," apparent circular reasoning.

Frances correctly indicated multiplication for gasoline consumption. On platinum she was allowed to go on because she could not figure out which operation to use. For the poster-picture, she was convinced that to decrease, divide was the correct operation: $1.2\overline{)0.052}$. (She subtracted on the written test.) Regarding the cookies, she selected 4 ÷ 16, and she seemed aware that each would get less than 1 kg of cookies. She translated into $4\overline{)16}$. She was aware that this answer would be 4. She feared that she had chosen the wrong one, but declined to offer another. For the paces, Frances chose multiply. When asked why she thought it correct she said ".74 m is one step. So, you have to times 9.2 m to find out how many steps . . .," a circular comment.

Shayne solved three of the five problems correctly. He used the rule that to multiply by a number less than zero [meaning one] the product will be less than the multiplicand. In the case of the platinum bar, he knew that the cost of 1 kg of platinum would be more than the cost of 0.75 kg of platinum. He said with assurance that it can't be "multiply [0.75 X $900] because multiply should make it smaller . . . . Divide, I think." He finally settled on addition "to make it larger." In the other case of the paces, a quotitive division situation, he chose multiplication, in spite of expecting the answer to be larger. Thus he deserted the rule that he had asserted and followed with confidence earlier.

Greg chose to multiply for gasoline consumption (correct) because he wanted the answer to be larger. For the platinum, he chose $900.00 ÷ 0.75 He said, "we want to know how much one is, so you divide by 0.75." For the picture, he multiplied first, but he placed the decimal point wrongly. He was asked if he might have made a mistake, but he was then convinced that he should divide, $1.2\,\overline{)\,0.052}$, "to make it smaller." When asked if multiply always makes it larger, he said, "No, . . . one times would be the same, or zero times would be lower." He chose division (wrong order) for dividing 4 kg among 16 people. He couldn't resolve the conflict of how the answer could be 4 kg, when we only had 4 kg to start. Regarding 0.74 m paces across a 9.2 m room, Greg first chose multiply because the answer would be larger than 9. When asked to justify his answer, he said " divide to find out how many." He also marked 9.2 ÷ 0.74. Greg appears to operate by the "divide makes smaller" rule. Had he gotten answers, however ,he might have adjusted, because he clearly had an idea of what to expect.

Jennifer knew that 1 kg of platinum would cost more than 0.75 kg. She was convinced that $0.75\,\overline{)\,900}$ was the answer, but her explanation was not clear. In the case of reducing a poster to a picture, a multiplication mistake made the answer too large. She then switched to division. She had not noticed that it should be reduced and "division makes it smaller." She knew that you should divide to distribute 4 kg of cookies among 16 people, but each could not get 4 kg (16 ÷ 4). She decided on "none of the above." For the number of 0.74 m paces across a room, she chose multiply. Her reasoning was circular: "It's multiply because . . . you multiply each step by the width of the classroom." Jennifer was capable of understanding some of the situations but was inconsistent.

Application of a Result

The last planned interview item was a 10 cm-by-10 cm grid with the problem, "Shade the diagram to show 0.3 of 0.2." The students had difficulty shading the diagram. With considerable prompting from the interviewer, all six were able to shade it, to make sense of unit, tenths and hundredths, and to determine that 0.06 of the unit was shaded. Each easily wrote an acceptable form of the equation 0.3 x 0.2 = 0.06. One requested, and others were prompted to reconsider the question, Multiply: .4 X .2 = ________, which had been the second interview question. All six changed their answer from 0.8 to 0.08.

DISCUSSION

There was no breakthrough in shading diagrams to show an area interpretation. Each student needed considerable prompting to complete the diagram. However, each target student had constructed a form of 0.3 x 0.2 that allowed application of that knowledge to the earlier problem. Each student quickly and confidently corrected the earlier mistake. This application provides the most solid evidence that the students "understood" multiplication of decimals, at least at a computational rule level.

Interestingly, all target students were able to choose the operation correctly in the gasoline consumption problem. Perhaps it was inherently easier because they knew the answer had to be more than 7.5 km, and both numbers are larger than one. It also appears that having a result helped to confirm the correct operation. When no result is obtained, as in the division questions, even good estimators will be unable to detect that a mistake has been made. In instruction, calculators are needed for testing many hypotheses and experiencing many results, without the distraction of laborious computation. As in this study, meanings or interpretations of many settings need emphasis. The students worked hard at making sense of (multiplication of) decimals. A review of the various concepts presented for exploration will be convincing evidence that it was not an easy task, nor was there sufficient time for reflection and complete concept construction. Children need many experiences to learn which properties of whole numbers generalize to rational numbers and which do not. Again calculators allow for the testing of hypotheses in making this construction.

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A PRELIMINARY REPORT ON THE RELIABILITY AND FACTORIAL VALIDITY OF THE RATIONAL NUMBER THINKING TEST IN THE REPUBLIC OF TRINIDAD AND TOBAGO

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ABSTRACT

This study, involving 481 Standard 3 through Form 5 (third-through eleventh-grade) students, is phase one of a major study involving about 6 thousand students conducted in the Republic of Trinidad and Tobago. It examines the reliability and factorial validity of the Rational Number Thinking Test and its four subtests (Ratio, Quotient, Operator and Measure). The data suggest that this instrument is an effective, valid and reliable means to identify student's perceptual/conceptual patterns and that students' approaches in dealing with the four suggested styles of presenting rational numbers are correspondingly clustered into four distinct cognitive manners.

STRUCTURE OF FRACTIONAL NUMBER PATTERNS

In trying to build fractional (rational) number experiences for children it is useful to try to understand children's mathematics (e.g. Harrison et al., 1980). One can think of fractional number knowledge for children not as fixed, but like other mathematics, the personal study of patterns whose meaning is tied up the whole personal and cultural experiences.

What is children's and young adults' fractional number knowledge like? Are there different pattens that comprise it? Can such personal knowledge be observed to 'contain' various patterns? What follows below is a brief review of a vision of ideal rational number knowledge. This structure contains four hypothetical intuitive knowledge subconstructs, which might form a transition between fraction related action patterns, such as partitioning and comparing to a dividable unit, and symbolic fractional number patterns. A brief description is given of the genesis of a paper/pencil instrument to observe these four patterns of fractional number behaviour. An analysis of the performances of 481 children age 7-15 in Trinidad is made to see if there are four distinct patterns as predicted and what the relation among these patterns might be.

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The Constructs

Some dozen years ago Kieren (1976) identified a number of mathematical/psychological subconstructs of fractional number knowledge, the central of which were ratio numbers, quotient numbers, operator numbers and measure numbers. Similar notions are discussed by Freudenthal (1983) in terms of fractional numbers as organizing various phenomena for young persons. Vergnaud (1983) summarised years of independent work coming to a similar conclusion and further saw rational number multiplicative structures as arising from quantity (scalar) and functional operators.

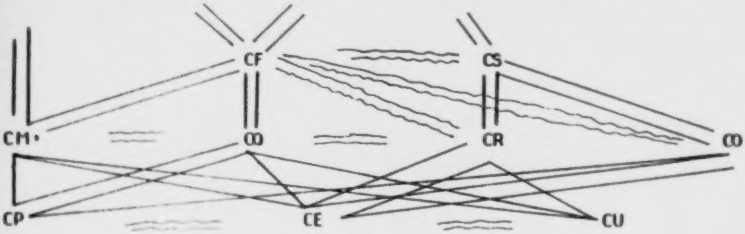

Figure 1

Kieren (1988) developed a scheme showing an ideal personal knowledge of rational numbers. A part of this scheme is shown in Figure 1. The four subconstucts CM+ (measure), CQ (quotient), CO (operator) and CR (ratio) are shown as deriving from mathematical action patterns or mechanisms of partitioning (CP), comparing with a dividable unit (CU) and equivalence (CE). These four intuitive subconstructs lead to higher level intuitive constructs of scalar and functional operators (CF and CS) an on to more symbolic and deductive knowledge patterns.

At a curricular level Behr, Post and Silver (1983) developed an elaborate model, materials and methods for giving children experiences with these constructs. Streefland (1984) also worked with children particularly using measurements, ratio and quotient experiences to identify their fractional thinking.

One can hypothesize these fractional knowledge building patterns as existing and possibly functioning as distinct thinking patterns in young persons. A complex of research questions are as follows:

Can such patterns be observed in young persons?

If so are they distinct or do they all form a single or perhaps dual fractional structure?

If they are distinct how can performances on the subconstructs be compared?

METHOD

Instrument

The Rational Number Thinking Test (RNTT) developed by Kieren was administered. It consists of 22 items in total. There are four subtests each presents positive rational numbers in a different style: 1) The chocolate-milk-mix drink (6 ratio number comparison), 2) the pizza sharing (9 quotient number comparison), 3) the machine (3 multi-part operator recognition) and 4) the shading (4 quantity construction) subsets. These items had come from extensive interview studies as well as using item structures from the work of Noelting (1980a & 1980b) and Noelting and Gagne (1980). The version of the instrument used here was redesigned tested and found reliable in 1984.

In a second run of the RNTT, the shading subtest was modified by adding seven more items presenting partitioning of two, three and four identical cakes in the shape of triangles, squares, regular pentagons, regular haxagons and regular octagons in addition to the original circular, subdivided rectangular and subdivided squared shapes. Only the triangular cakes were presented into two forms, one with subdividing line segments and the other without. Students' partitioning patterns this may reveal await another apportunity.

Subjects

For this report, the subjects involved were 481 students in Standard 3 through Forms 5 i.e. grades 3 through 11. They were randomly picked from about 6 thousand students in Standard 3 through Form 6 (grades 3 through 12) involved in a major research project (Rahim, 1986) sponsered by the University of the West Indies, Trinidad. For this paper the following hypothesis is considered: Is there a factor structure of the correlations which shows the items of the four subconstructs to be distinct or orthogonal?

General Procedures

Students were administered the RNTT by thier classroom teachers. The RNTT was prepared in two forms, FORM A and FORM B. Either form was administered over two separate classroom sessions, the first for subtests 1 & 2 and the other for subtests 3 & 4. In research-teacher conferences, the participating teachers were briefed about the content of the RNTT, informed and provided with instructions.

After developing a marking scheme with over .95 enter-rates reliability, the items were marked using a scheme from Noelting and Gagne (1980) on the comparison items and a simple right/wrong scheme on the recognition and construction items.

Estimates of internal consistency (Cronbach's alpha coefficients) were calculated for each of the four subtests.

A 31x31 matrix of item intercorrelations was formed. A principal component factoring analysis of the data and a classical factor analysis with a four factor solution and varimax rotation were performed.

RESULTS

The factor loads for the four factor solution with values above or equal to .45 are presented in Table 1

The Cronbach's alpha coefficients were .85, .86, .88, and .86 for the chocolate milk-mix, pizza, machine, and shading subtests respectively

TABLE 1

Varimax Rotated Factors for the Four RNIT Subtests

Subtest	Item #	Factor I	Factor Loading Factor II	Factor III	Factor IV
Chocolate	1				.499
	2				.744
	3				.765
	4				.674
	5				.602
	6				.631
Pizza	7	.628			
	8	.673			
	9			.521	
	10			.638	
	11			.710	
	12			.629	
	13			.705	
	14			.759	
	15			.745	
Machine	16a	.551	.454		
	16b	.453			
	16c	.595	.537		
	17a		.777		
	17b		.616		
	17c		.782		
	18a		.819		
	18b		.603		
	18c		.806		
Shading	19	.522			
	20a	.640			
	20b	.664			
	21a	.681			
	21b	.702			
	22a	.727			
	22b	.708			

The percentages of common variance accounted for each factor were 30.588%, 25.629%, 23.110% and 20.674% resepctively The percentages of variance accounted for by each factor were 18.099%, 15.164%, 13.674% and 12.233% respectively, i.e. the four factor solution accounted for 59% of the total variation. The eigenvalues of the first four factors from the principal component factoring were 12.41, 2.54, 1.94 and 1.47 respectively.

Principal component factoring analyses for the portions of the data of the elementary and the secondary levels were separately conducted. The results for either one were parallel to the above overall factoring analysis.

DISCUSSION

The findings of the factor analysis and the reliability coefficients of the four RNTT subtests suggest that the scores of the four subtests are sufficiently stable to be used as separate scores. The factor analysis (Table 1) shows that the four subtests share an extremely small amount of common variance; only two items of subtest 2 (pizza) and one item of subtest 3 (machine) are clustered with the items of subtest 4 (shading) on Factor I. This indicates that, for grades 3 through 11 (Standard 3 through Form 5), the subjects' cognitive approaches for these four styles of presenting rational numbers are quite distinct.

Each subtest usually takes less than fifteen minutes to administer, it would therefore be convenient to use by teachers and researchers to document student's conceptual and/or perceptual patterns, imagery, the mathematical language students are using and various student's misconceptions. This would help teachers and researchers to identify aspects of a student's knowledge building and to plan appropriate experiences that can help students build knowledge.

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REPRESENTING AND SOLVING MULTIPLICATIVE STORY PROBLEMS AS PROPORTION PROBLEMS WITH A UNIT RATIO - A TEACHING EXPERIMENT

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ABSTRACT

This paper reports the results of a study designed to test the effectiveness of an experimental solution strategy for solving one-step multiplication and division story problems with non-routine numerical characteristics. The strategy used a data table representation and proportional reasoning to guide students' thinking. Results indicated a statistically significant difference in favor of the experimental strategy as compared to the traditional strategy of using simpler numbers.

INTRODUCTION

Recent studies have demonstrated that many students, in writing an arithmetic sentence to represent the solution to a multiplication or division story problem, have difficulty choosing the correct operation or ordering the terms correctly in a division equation when the numbers in the problem are non-routine. In multiplication, for example, a multiplier less than one has been shown to reduce success rates considerably (Greer & Mangan, 1986). In division problems a decimal divisor or a divisor greater than the dividend also has proven to negatively affect success (Bell, Fischbein, & Greer, 1984; Bell, Swan, & Taylor, 1981).

It has been hypothesized that this lack of success is due to the student's attempt to fit the givens of a particular problem into the behaviorally based concrete models of repeated addition for multiplication, and partition and quotition for division, and to the student's incomplete understanding of these operations, that is, that multiplication always makes larger, division always makes smaller, and division by a larger number is not permissible. These models and understandings constrain or restrict the student's thinking. Therefore the student may reject multiplication as the correct operation for a multiplication problem in which the multiplier is a decimal because the numerical characteristics violate the "always makes larger" understanding of multiplication and the repeated addition model. Similarly, the student may reverse the terms of a division problem when the divisor is larger than the dividend.

RATIONALE

The experimental instructional strategy tested in this study was based on the following hypotheses and findings from research:

1. The ability to represent a problem situation in different modes and to translate between the different modes is basic to successful problem-solving (Kaput, 1987).

2. All simple one-step multiplication and division problems which fall under the general heading of rate problems can be expressed as simple direct proportion problems with a unit ratio (Vergnaud, 1983).

3. There is some evidence that schematic representations of problem components facilitate understanding and correct solution strategies (Willis & Fuson, 1985).

4. Proportional reasoning has been observed in the solution strategies of younger children on proportion problems with simple numbers (Karplus, Pulos, & Stage, 1983; Tourniaire, 1986).

The experimental instructional strategy avoided the use of the concrete models and the incomplete understandings of multiplication and division by introducing an intermediate representation between the verbal representation and the purely symbolic representation of an arithmetic sentence. This representation was a schematic representation in the form of a data table called a linking table. Students were instructed to construct a table from the problem givens, the general form being

(Label)	(Label)
a	b
c	?.

As a rate problem, one of the quantities in the table was always a unit quantity. Students were then taught proportional reasoning, that is, to find the multiplicative relationship between the unit quantity and another quantity in the table and use the same relationship between the third quantity and the unknown quantity to find the unknown. For example, if $a = 1$, students were taught to determine the rule linking a to b ("times b") and to use the same rule to link c to the unknown, resulting in the equation $c \times b = ?$.

METHOD

The effectiveness of the strategy was tested against a commonly used strategy of substituting simpler numbers in the problem to determine the correct operation. Five intact classes of average- to above-average-ability seventh-graders participated in the study. Two units of instruction of ten lessons each were developed, one teaching the experimental strategy and one the traditional strategy. The five classes were randomly assigned to either the experimental or control group (resulting in three experimental and two control classes) and taught the corresponding unit by the regular teacher. A pre-test of general mathematics ability was administered and results indicated no significant difference between groups on that measure. A posttest measuring the subjects' ability to write the correct arithmetic sentence for multiplicative story problems containing non-routine numbers was administered. The posttest consisted of 18 problems, 9 of which had numerical characteristics which violated the constraints described earlier. Table 1 summarizes the numerical characteristics of those nine problems. A factorial analysis of variance with treatment, gender, and mathematics ability level as the independent variables and posttest results as the dependent variable was performed. See Sellke (1988) for details.

RESULTS AND DISCUSSION

The statistical test indicated a significant difference for the main effects of treatment ($p < .001$) and ability level ($p < .05$), no significant difference for gender, and no interaction. Only the treatment main effect will be discussed here.

Table 2 summarizes the mean scores and percentages for the two treatment groups on the posttest. What is most striking is the high level of success on the part of the experimental group. Even the low ability students were able to achieve at a high rate (81%), a rate even above the level achieved by the high ability students of the control group (69%). It is conjectured that the linking table strategy was successful because it used a schematic representation which was meaningful (the linking table), stressed relationships between quantities rather than concrete action on a single quantity, and based the arithmetic sentence-writing task on relationships explicit in the table rather than behaviorally based models and incomplete understandings of the operations.

The success of the linking table strategy suggests implications and further research in other areas of the mathematics curriculum. Besides introducing multiplication and division to young children on the basis of the concrete models of repeated addition, partition and quotition, introducing these operations in terms of data tables and multiplicative relationships between quantities of the tables will give these students another, more mathematical, way to begin to understand them.

Table 1. Numerical and Contextual Characteristics of Posttest

Multiplication		Partition		Quotition	
Mld	Mlr	Dvd	Dvr	Dvd	Dvr
11	0.78	16	43	21	34
2.2	0.66	0.06	4.8	0.15	0.42
0.45	0.37	0.38	0.83	0.12	3.6

Note. Mld = multiplicand; Dvd = dividend; Dvr = divisor.

Table 2. Cell Means (Percentages) for Posttest Problems

Total	8.00 (89)	5.05 (56)
Ability		
Hi	8.67 (96)	6.20 (69)
Ave	8.06 (90)	4.56 (51)
Lo	7.25 (81)	4.18 (46)
Male	7.77 (86)	5.75 (64)
Female	8.27 (92)	4.77 (53)

Note. Maximum possible score = 9.

This initial understanding of multiplication and division as operations will provide groundwork for later use of the linking table strategy when the

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concrete models become inadequate representations due to non routine numerical characteristics. Applications of the strategy to other rate situations, such as percentages, scale drawings, and measure conversion also hold promise. Finally, employing the strategy to understand and solve proportion problems of a more general nature may also help students to understand and use true proportional reasoning earlier and to a greater degree than they now do.

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SOLVING STORY PROBLEMS INVOLVING DIVISION WITH REMAINDERS: THE IMPORTANCE OF SEMANTIC PROCESSING AND REFERENTIAL MAPPING

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Students' ability to solve division story problems involving remainders was examined. Embedding a target divison problem involving remainders in a small set of related problems enhanced student performance. The enhanced performance is explained in terms of students engaging in relevant semantic processing and mappings between and among three referential systems: the story text, the story situation, and the mathematical model.

Remainders are interesting mathematical entities that elementary school students encounter first in the context of whole number division and later in the context of rational numbers. Students typically learn to express remainders in several different symbolic formats. For example, the result of 50 ÷ 4 might be expressed as 12 R2 in third grade, as 12 1/2 in fifth grade, and as12.5 in seventh grade. Although students generally have little difficulty with these forms of expression for remainders in routine computation settings, they often experience considerable difficulty when the computation is embedded in a problem situation.

When one considers "real life" situations, the source of the difficulty becomes clear, since the same symbolic expression of a division problem can represent very different problem situations and have different answers that depend upon important aspects of the situational context and the quantities involved in the problem. For example, the indicated division "100 ÷ 40" is suitable to help answer each of the following problems: "Mary has 100 brownies which she will put into containers that hold exactly 40 brownies each. (1) How many containers can she fill? (2) How many containers will she use for all the brownies? (3) After she fills as many containers as she can, how many brownies will be left over?"

To solve each of the above problems, a person would perform the same calculation, 100 ÷ 40, but the answer to each problem would be different. The correct answer for (1) is 2, since only two containers would be filled by the 100 brownies. In this case, the remainder is essentially ignored. The correct answer for (2) is 3, since a third container would be needed for the "extra" brownies that are not in the two full containers. In this case, the existence of a remainder leads us to increment the quotient when answering the question. It is interesting and important to note that the correct answer does not occur explicitly in the computation; the answer is constructed from data supplied both by the mathematical computation and the story situation. The correct answer for (3) is 20. In this case, the correct answer is the remainder.

Unlike most story problems that students solve in elementary school, these problems do not allow the mindless computation of answers without semantic processing. Their successful solution depends to a great extent on mappings between and among at least three referential systems: the story text, the story situation, and the mathematical model. The distinction between the story text and the story situation, which has been articulated by Kintsch (1986), may not be relevant to much of the problem-solving activity observed when elementary school students solve typical textbook story problems, but it is of critical importance in the solution of division story problems involving remainders.

In order to solve the brownie problems successfully, one might map from the story (natural language) text representation of the problems into a mathematical model representation of 100 ÷ 40, then perform the indicated computation within the referential system of mathematics, expressing the resulting answer with an appropriate mathematical representation, then map that result back either to the story text representation or to the implied story situation (in the "real world") representation in order to decide how to treat the quotient and remainder, and finally obtain suitable mathematical and natural language representations of the solution that have accompanying interpretations and validity within the referential systems of real world situations and the knowledge domain of mathematics.

The widespread failure of students to succeed in solving problems involving whole number division and remainders has been documented through the National Assessment of Educational Progress and several state assessments. For example, only 24% of a national sample of 13-year-olds was able to solve correctly the following problem which appeared on the Mathematics portion of the National Assessment of Educational Progress in 1982 (NAEP, 1983): "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?" Similarly, the following division problem, which appeared on the California Assessment Program Mathematics Test at Grade 6, was answered correctly in 1983 by only about 35% of the sixth grade students who took the test: "The 130 students and teachers from Marie Curie School are going on a picnic. Each school bus holds 50 passengers. How many buses will they need?" In both cases, students often erred by choosing answers that contained decimals or fractions (e.g., 2.6 for the school bus problem).

An earlier study (Silver, 1985) explored several alternate versions of the school bus problem that were aimed at increasing the likelihood that students would attend to relevant information in the problem. The results of that study suggested that variations in the problem, such as making explicit certain implicit information in the problem or the required solution, could significantly enhance students' performance.

Unlike most considerations of "relevant information" for problem solving in elementary mathematics, the focus of attention here was not so much on information that would enhance the mapping between the story text and the mathematical model but rather enhance the mappings between and among these two reference spaces and the story situation. Therefore, one interpretation of these results and the assessment results is that students are incompletely mapping among the relevant referential systems. In particular, they appear to map from the problem text to a mathematical model (in this case, a computation to be performed), compute an answer within the mathematics domain, and fail to return to the problem story text or to the story situation referent in order to determine the best answer to the question.

In the current study a somewhat different approach was taken to increasing the likelihood that students would engage in the referential mapping necessary to obtain a correct solution. This study examined the effects on student performance of prior problems requiring the same computation and similar referential mappings.

METHOD

Subjects

The subjects were 141 eighth-grade students attending an urban middle school.

Design

Each subject completed a paper/pencil test containing three problems. The problems used in the study are found in Table 1. Problem C was the target problem of interest, problems A and B are the closely related problems, and problems X and Y were used as "filler" problems to keep all versions the same length. Each subject was randomly assigned to one of four alternate versions: ABC, ACX, BCX, CXY.

Table 1

Problems Used in the Study

The science teacher at Marie Curie School has been given 730 frogs. The frogs will be kept in tanks. Each tank holds 50 frogs.

(A) How many tanks can be completely filled with frogs?

(B) If all of the tanks used are full of frogs, how many frogs will be left over?

(C) How many tanks are needed to hold all of the frogs?

(X) If the average frog is 50 millimeters long, about how many centimeters long are 6 frogs? (1 cm = 10 mm)

(Y) The covers on four full tanks were left off and some frogs escaped. There are now 186 frogs in those tanks. How many escaped?

(1) 14 (2) 14 3/5 (3) 14 R30 (4) 15 (5) 30

RESULTS

Target Problem C was the most difficult problem when it appeared in the first position on a test form. Overall performance on Problems A, B, and C, independent of position in the test form, was generally poor (43%, 55%, and 48%, respectively). Performance on target problem C was significantly better on versions ABC (66%; $t = 4.1$; $p \le .0005$), ACX (57%; $t = 3.2$; $p \le .005$), and BCX (43%; $t = 1.8$; $p \le .05$)

than it was on version CXY (22%). As the above data suggest, problems A and B had somewhat different effects on performance on Problem C. A direct comparison of performance on Problem C in versions ACX and BCX revealed differences in favor of version ACX, although the differences were not statistically significant (t = 1.2; ns).

DISCUSSION

The results indicate that student performance on the target problem was enhanced by having students also solve relevant, related problems. How can we explain this improvement?

Perhaps the improved performance was simply due to students' recall of solutions from prior problems. However, since the related problems had different solutions, the enhanced performance could not be due solely to recalling the solution of the earlier problems. Perhaps the experience with the calculation involved in the prior problems is the source of the enhanced performance on the target problem. The substantial, although not statistically significant, differential effects of Problems A and B would appear to refute the claim that the improvement on Problem C was due simply to students having performed the required arithmetic calculation in a prior problem. Since Problems A and B both required the same calculation as Problem C, both problems would be expected to have the same effect if the effect were due simply to the prior calculation.

The results are consistent with the explanation that enhanced performance is due to the students' increased sensitivity and attention to the relevant semantic and referential mappings involved in the target problem solution. If students who solved Problems A and/or B gave increased attention to relevant mappings within the referential space for Problem C, then one would expect their performance to be enhanced. In particular, experience with Problem A or B might draw attention to the need for mapping into either the story text representation or the story situation representation after obtaining a solution through use of a mathematical model.

The differential effects of Problems A and B are somewhat more difficult to explain. The referential mappings for Problems A and C are clearly isomorphic, and the expected strong effect was found. The relationship between Problems B and C is somewhat less clear. It might be possible that one could obtain a correct solution to Problem B without mapping back into the story situation after completing work with the mathematical model. If so, then Problem B would be expected to have a weaker effect on performance on Problem C. Alternatively, it may be that the difficulty of Problem A contributed to its enhanced effect, since the literature on information transfer suggests that it is easier to get positive transfer between problems if the first problem is more difficult than the second. These competing explanations, and other related questions, are being examined in further data analysis and in subsequent research which studies the mutual effects of the three related problems.

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TWO STUDENTS; TWO REPRESENTATIONS FOR FRACTIONS

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This paper compares two eighth grade students' strategies for partitioning paper strips to make "fraction rulers", their use of number lines in making decisions about fraction order, and their attitude toward this kind of work. In spite of similar ability to order fractions, the students' preferred representations differed greatly. One student used mental images of the strips to estimate answers to simple computation problems and to order fractions while the other student used numerical procedures he had memorized.

Background

The students, Matt (age 15) and Sophie (age 13) were in a below average eighth grade mathematics class. During the previous year, they had been subjects in a ten-lesson teaching experiment designed by the Educational Technology Center (ETC) fractions group (Davidson, Dickenson, & Tierney, 1986), which used number line presentations to develop children's conceptions of fraction order. In both the ETC work and the extension of that work reported here, the creation of fraction rulers from paper strips was a central task.

In the ETC work, both students had been conscientious in trying to mimic the researcher's actions in folding paper strips into fractional parts and labeling them to use as rulers in measurement activities, but neither understood the intention of these actions or learned from them much about the order of fractions. Matt had begun the strip-labeling with 1/n instead of 0/n, writing for example 1/8 instead of 0/8 at the left end and 9/8 at the right end of the eighths strip, and Sophie attempted to partition strips into eighths and tenths by folding over and over and then opening and counting the intervals without making sure they were equal. In interviews seven months later at the beginning of the current study, neither student was sure whether one third was larger or smaller than one fourth. Their ability to order pairs of fractions developed at nearly the same pace during our work together. Both students quickly created ways to compare fractions which had either the same numerator or the same denominator and by the end of this work, both used one half and one whole as reference points in comparing fractions.

Methodology

We used a more constructivist approach in the second year of ETC research and in the work with Sophie and Matt discussed here. To allow for a number of different strategies, we presented the tasks as questions to answer or problems to solve. In twice weekly sessions over a five month period, Sophie and Matt, sometimes alone and sometimes together, worked on a variety of tasks which included partitioning paper strips and labeling them to use as rulers, estimating fractional lengths in board games and on computer-generated number lines, ordering fractions written on cards, and addition and subtraction of fractions. The focus of this report is on how the students' quite different representations affected their work with the strips: the attributes of the strips they attended to, their strategies for partitioning the strips, and their use of the strips in other activities.

I presented the task of partitioning the paper strips quite differently from the ruler-making of the previous year: If an ant was walking along the top edge of the strip, where it would be when it was one third, two thirds, three thirds of the distance across the strip? (from Labinovicz, 1985). The students marked seven ten inch strips: one in each denominator to tenths, omitting sevenths and ninths. After marking each strip, they checked the accuracy of their placement of the marks against a pre-made template and made corrections. They then could use their accurately marked strips in marking strips for fractions of higher denominators. I did not ask them to label the marks with 1/3, 2/3, etc., but left it up to them to label them as they wanted or not at all.

While they were working, I interviewed the two students in a style described by Eleanor Duckworth as "research as teaching" (1987). By asking them to explain their thinking and by adjusting the tasks to challenge their thinking, I sought to extend their knowledge at the same time as I learned about their strategies.

Results

Matt made decisions about where to place divisions on the paper strips without actually looking at the strips. He relied on his memory of relationships between fractions, such as one fourth is half of a half, or he computed equivalents on paper to help him decide where a fraction belonged. In this way he placed 2/6 and 4/6 on his sixths strip approximately where the 1/3 and

2/3 marks were on his corrected thirds strip. However, when he placed the marks according to these number relationships, he seldom checked to see that the placement made sense or that the divisions were equal. An erroneous idea he held that one-third is half of a half put 2/3 at the half and 3/3 well before the end, and he tended to place marks to the left of where they belonged, leaving intervals toward the right end of the strips much larger than those near the left end.

When Sophie in this second year no longer thought she was expected to fold the strips as the teacher did, she kept the strips out flat and planned the placement of the marks so the intervals would be as equal as she could get them. She invented a method of partitioning the strips by eye into unit fractional parts (1/n) and iterating, using her fingers as calipers, to find multiples of these. She would adjust the single unit until n/n exactly filled the strip or, if this proved difficult, she would tentatively mark all of the divisions at the same time by laying pencils and pens across the strip and adjusting their placement until she was satisfied that all the intervals were the same size. With these methods, she could partition a paper strip or name points along number lines in computer and board games with extraordinary accuracy. Sophie did not write numbers on the strips but instead counted intervals to find the distance of the marks from the end of the strips. Her knowledge of equivalents came later than Matt's. First she memorized arrangements of numbers on the strips. Thus, she noticed that the marks for two-fifths and three-fifths were symmetrically arranged around the mark for one-half whereas the mark for three-sixths was in the same place as the mark for one-half.

These ways in which Sophie and Matt approached the task of partitioning the strips fit with the representations for fractions which they talked about and used in other activities (Tierney, 1988). Matt consistently spoke of fractions as the numbers themselves and fraction learning as the memorization and application of the numerical algorithms teachers teach. To compare fractions, he tried to use common denominator strategies which he had learned in his regular math class. Matt implied that Sophie's facility with pictures was less important than knowing procedures with numbers. He said that, when he and Sophie started working together, he knew more "rules" about fractions than she did and that his teacher never used pictures for fractions. Although Matt didn't like the games we played because he "could never win", he used them as tools to clarify some fraction order relationships he had noticed previously. Imagining where the ant would be on the number line made him aware that one could think about fractions as more than just numbers.

Although Sophie would sometimes call out numbers almost randomly as answers, apparently because she thought numbers were expected, she usually spoke of fractions as fractional parts of cakes or of sets of cookies, and in the process of this work she came to represent them as portions of the paper strips. She spoke often of "just seeing" relationships between various lengths. To compare fractions, she drew and partitioned parallel line segments or referred to her mental images of partitioned paper strips. These continuous linear models she developed as her representations for fractions allowed for a qualitative sense of size. She spoke of one fraction being "a tad bigger" than another and explained that unit fractions with larger denominators are smaller than those with smaller denominators "because they are closer together" on the strips. She usually avoided computation by getting up from the table or excusing herself from the room when Matt asked for help with their homework. When I insisted she try some addition of fractions, I discovered that if the numbers were small she could estimate by bringing visual images of the strips to mind (2/3 + 1/2 = 1 1/6 because two-thirds is just a little longer than one-half and one-half and one-half is one and a little more, about a sixth more). When she was unable to create a mental representation for the numbers, she adjusted the algorithm so the problem made sense to her (1/5 + 1/17 = 4/20 +1/20 because five goes into twenty four times and seventeen goes into twenty once.)

Both Sophie and Matt had difficulty handling tasks which were presented in a form which was different from their preferred representation. Matt looked for information only in numbers. On a board game in which players were to estimate distances from point to point in terms of parts of strips, Matt said he could not see how anyone could make a guess. He wanted to hold a labeled fractions ruler against the mat to determine the length he needed. When that was not allowed, he limited his guesses to one-half or one-whole. Where strips had both marked divisions and numbers, Matt worked with only the numbers. At times the written numbers became for him what the Rational Number group call "visual distractors" (Post et al, 1985), as when he thought a paper strip was only two thirds of a strip long because 2/3 was the largest fraction written on the strip. He used five and one third strips to measure out what he thought was a length eleven-third strips long (two thirds from each of the five whole strips and one third more). With some probing, he decided that 3/3 could be written at the end of the strip, and was able to do the problem correctly even with the strips turned over so that no marks showed. He did this, not by actually visualizing the three thirds on the strip as Sophie would do, but by remembering that there should be three thirds for each strip.

Because written and spoken numbers are the language in which problems are posed and answers are expected, Sophie had to deal with the numbers. However, it was the visual images from which she got the most information. After working for a few sessions partitioning the strips, she began to translate numbers to mental images of the strips, operate on those images to get the answers she wanted, and then translate her results to numbers to communicate her results. When both numbers and visual cues were available, she tended to ignore the numbers. Because she figured each guess separately in games, she was not aware of equivalent fractions. (Once when Matt suggested guessing one half, she said she thought five tenths would be better.) In all of her work partitioning the strips and as much as possible in games with number lines, her attention was on the intervals rather than on the endpoints of intervals or the written numbers with which they were sometimes labeled. To measure out eleven-thirds strips, Sophie touched each interval of her strip partitioned in thirds as she counted "one, two, three," and then on other strips which she laid out with the writing face-down, she touched imagined intervals while counting "four, five, six, ..." until, touching the fourth strip twice, she reached "ten, eleven".

The distinction between Matt's preference for numbers and Sophie's preference for visual images was apparent again toward the end of the study when they were to sort fractions smaller, larger, or equal to one half and think of some additional fractions which belonged in these categories. No particular representation for one half was suggested so the students had to provide their own. Matt considered the numbers . He set up the problem as comparing the numerator to half of the denominator. He did this quickly if the denominator was an even number but made mistakes halving odd denominators. Sophie could not at first figure out how to recognize a fraction equal to one half, but when I asked her how she would know whether she had more or less than half of something, she developed a discrete representation, more precise than her usual continuous one, but still a visual image. She imagined sets of cookies and compared the cookies taken (numerator) to the cookies remaining (difference between the numerator and the denominator). Matt found it much easier to find additional fractions equal to one half than to think of those smaller or larger than one half. He wrote the series which was familiar to him from class: 2/4, 3/6, 4/8, etc. The fractions he thought of for the other two categories were mostly those with which he was also familiar: unit fractions (1/3, 1/4, etc.) for smaller than one half and 3/4, 5/6, 7/8, etc. for larger than one half. Once she had developed her strategy, Sophie found all three categories equally easy to fill and she thought up what Matt called "hard" fractions such as 2/6, 50/100, and 9/6.

Discussion

Because the students resisted translation from their preferred representations, they were limited in the kinds of problems they could solve. In our work, Sophie fared better than Matt. Her visual approach was more efficient than his still quite limited use of equivalents. The use of strips and the number line games opened up for her a whole new view of fractions. However, although she could estimate, she could seldom get exact answers. She could not or would not memorize procedures which she didn't understand and thus was unable to do her homework and barely passed her regular math course. Matt, who completed all work on time, fared better in his regular math class, getting Bs. While he made many common mistakes such as adding denominators, he was far more successful than Sophie at mimicking the procedures the teacher demonstrated. However, he had no visual images with which to estimate or to check his computation.. He could compare fractions using equivalents to one-half ("I can see the relationship between seven-tenths and three-sixths because five-tenths is one-half and seven is more than five") and yet miss seeing relationships between fractions which were further apart ("but I can't see the relationship between seven-tenths and one-third").

Matt was like many of the seventh grade students in the ETC work who tried to memorize what they were taught in class. Sophie was more like the fifth grade ETC students who came to fractions without a repertoire of procedures learned from teachers and thus had to be more inventive in their solutions. If we continue, through the work we assign and the ways we measure achievement, to reinforce the idea that only number work is important and certain procedures must be memorized, we penalize both groups. Those like Matt lose because they have no reason to develop their own representations which could provide meaning and a check for computation. Those like Sophie lose because their skills in visualization and estimation are not appreciated in their regular math class so they are not challenged to connect them to the computational algorithms they see there. All students lose because they are not challenged to create their own mathematics and to think of themselves as mathematicians who can figure things out for themselves.

Hasemann (1985) describes two students with styles of mathematical thinking similar to Matt's and Sophie's, who, like them, had not yet related different aspects of the concept of fractions. I concur with his conclusion that we must begin with the children's individual ideas rather than those of teachers or textbooks. We must provoke children to construct their own representations and to expand these to solve problems which are presented diagramatically as well as those which are presently numerically. After the students have their own solutions, traditional algorithms can be shared as some possibilities among many.

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NUMERICAL APPROACHES TO ALGEBRA PROBLEM SOLVING IN A COMPUTER ENVIRONMENT*

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This paper reports the findings of a study carried out during the past year on the guess-and-test numerical methods used by pre-algebra students in a computer problem-solving environment. The main focus of the analysis was the strategies used when input and output values are displayed in tabular form. It was found that subjects of the study tended to use increasing strategies, even in problem situations requiring a decreasing strategy. No difference was found between approaches used in problems with known context and problems where the underlying situation and problem calculations were hidden. Within the context of a tabular representation of input and output values, subjects used non-contextual processes based simply on order relations and number patterns.

At the recent Research Agenda Project conference on the learning and teaching of algebra, Fey (in press) argued strongly for radical changes in the goals and teaching of algebra. He pointed out that the availability of computer technology offers access to entirely new kinds of important algebraic problems and methods. One of the methods that he suggested will be used extensively in future technology-rich algebra curricula is that of successive approximation. Most successive approximations begin with choosing some initial value (often a guess), carrying out a test, and revising the approximation in some systematic way. With computers to speed the test phase, such methods can become very effective.

However, it is not merely the presence of computers that makes guess-and-test methods attractive. Another seeming advantage of these methods is the opportunity they provide to link algebra problem solving with students' prior arithmetic knowledge and experience. The substitution of numerical test values into algebraic representations of problems permits the beginning algebra student to construct meaning for problem representations that may be different from those used in the past, while at the same time using solving methods based on familiar arithmetic techniques. But, as Fey states, "we know very little about the ease or difficulty of

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developing in students a facility for trial and error methods" in computer environments.

This issue, and others, are being investigated in a multi-phase research project on the use of computer methods in the teaching and learning of algebra. At the PME-XI conference last year in Montreal, we described the theoretical foundations of this project (Boileau, Kieran, & Garançon, 1987). Our paper this year reports the results of a study carried out during the past year on the guess-and-test numerical methods used by pre-algebra students in a computer problem-solving environment.

The Study

Computer Environment

CARAPACE is a triple-mode computer problem-solving tool. The first mode allows the user to enter an algorithm for solving a problem. This algorithm is to be specified in terms of input variable(s), a procedure for calculating the solution, and output variable(s) (see Figures 1 and 2). Herein lies the algebraic component of the software: having to specify an algorithm for solving a problem in terms of the problem variables.

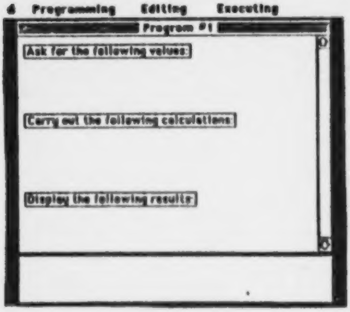
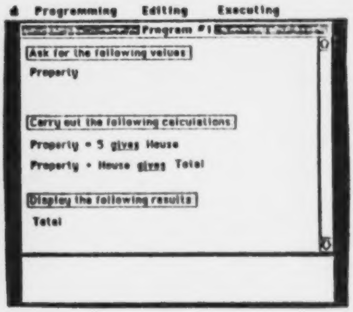

Programming
Editing
Executing
Program #1
Ask for the following values:
Carry out the following calculations:
Display the following results:

Figure 1. CARAPACE Computer Screen beforeuser enters an algorithm for solving a problem.


Programming
Editing
Executing
Program #1
Ask for the following values:
Carry out the following calculations:
Display the following results:

Property

Property * 5 gives House

Property + House gives Total

Total

Figure 2. CARAPACE screen after user has entered an algorithm for the problem: "Johanne's house and property together have a value of 76 950$. The house has a value of 5 times the property. What is the value of each?"

The second mode of CARAPACE requires that the user provide a value for the input variable(s) and then shows the step-by-step calculation of all the intermediate values resulting from that particular input (see Figure 3). For each subsequent value tested, the screen is cleaned and shows the calculations carried out in arriving at the

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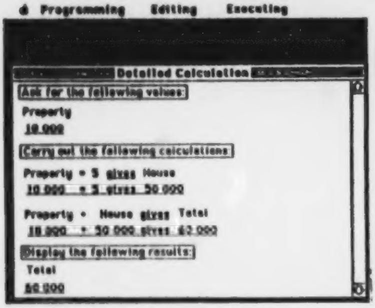
result for that test value.

The third mode of CARAPACE--the crucial one for this particular study--presents a tabular display of input and output values. Figure 4 indicates an example of a tabular display after the user had tested five different values for the input variable, Property. Note that, while in this mode, the user is not presented with any intermediate calculations. The user simply types in a numerical input, which appears on the screen in the left-hand column, and the computer displays in the right-hand column the corresponding value of the output variable.



Ask for the following values:

Property
10 000

Carry out the following calculations:

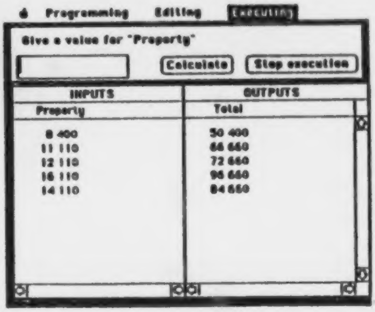
Property * 5 gives House
10 000 * 5 gives 50 000

Property + House gives Total
10 000 + 50 000 gives 60 000

Display the following results:

Total
60 000

Figure 3. Display of the step-by-step calculations carried out by CARAPACE for the algorithm displayed in Figure 2, after the user had provided the value 10 000 for the input variable Property.



Give a value for "Property"

INPUTS	OUTPUTS
Property	Total
8 400	50 400
11 110	66 660
12 110	72 660
16 110	96 660
14 110	84 660

Figure 4. Tabular display of first 5 trial values provided by user for problem of Figure 2. Target value for the variable Total is 76 950.

Types of Problems Used

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1. The town of Verval has twice the number of inhabitants as Beaubourg, and Beaubourg has 87 654 inhabitants fewer than Montclerc. If the total populations of these three towns is 567 890, what is the population of each town?
2. The number of students in your school is 15 times the number of teachers. Altogether the number of students and teachers is 256. How many teachers are there?
3. A town is inhabited with 256 434 people. Because of pollution, many people leave town to live in the country. Each year the population decreases by 1623 persons. In how many years will there be only 128 217 people left in this town?
4. You would like to buy a pair of jeans and a t-shirt. Together these two items cost 77,25$. The difference in price between these two items is 34,25$. What is the cost of the jeans?
5. Find two numbers whose sum is 42 and whose product is 377.

Figure 5. Examples of some of the problem types used in the study .

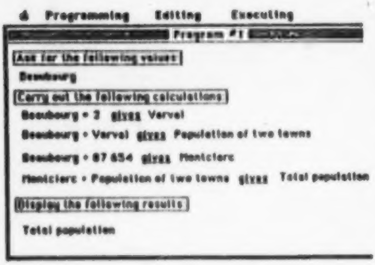
Most of the problems used in the study were linear, in one or two variables (e.g., the first four problems in Figure 5). The first two of these directly lend to representation as increasing functions (i.e., as the input increases (or decreases), the output increases (or decreases)), the third as a decreasing function (i.e., as the input increases, the output decreases), and the fourth directly lending itself to neither increasing nor decreasing representations. Variations on these problem situations involved different sizes of numbers, including decimal and non-decimal examples. Two-variable, non-linear problems were also included (e.g., the fifth problem of Figure 5).

Research Procedures

The children who participated in the study were two thirteen-year-olds considered by their teacher to be of average to upper ability in school mathematics. Each child worked individually on a set of problems during a 60-90 minute session each week for a total of ten weeks. One researcher was responsible for interacting with the children throughout the ten-week period, while the other researchers observed, took notes, and handled the video equipment.

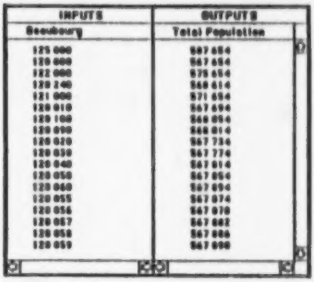
Procedure (a). The children generally began to solve each problem on paper. We encouraged them to use guess-and-test methods by asking them questions such as, "Is it possible for the population of Beaubourg (first problem of Figure 5) to be 1 000?" Following interventions of this type, one of the children first tried 100 000 for Beaubourg, which yielded a total population of 487 654 for the three towns. His next trial value was 150 000, which yielded a total of 687 654. He then went to the computer and entered the program shown in Figure 6, receiving help, whenever necessary, from the interviewer. Next, he decided to go directly to the third mode of CARAPACE, the tabular display--bypassing the second mode that provides the detailed calculation of intermediate values. His first trial value with the computer was 125 000--a value midway between the two pencil-and-paper values he had tried. After entering 125 000, the computer displayed 587 654 in the right-hand column (see Figure 7). He continued with different trial values--explaining for some of these values why he was choosing them--until he came up with the one that yielded the target population of 567 890. It should be noted that no help was provided to the children while they worked in the tabular mode. Furthermore, for some of the problems, the children did not need to work with the computer; they were able to solve the problem with only a few pencil-and-paper trials.

Procedure (b). Another research procedure used was one that involved only the tabular display mode. Subjects were merely told what the target output of the "hidden problem" was. They had to generate trial values permitting them to reach that target, without knowing any of the underlying calculations. Each value tried, along with the result of that trial, was displayed one-after-the-other in the table. This "hidden problem" procedure was used for the last few problems of each session. It was hypothesized that the children would use different approaches in generating trial values for problems with known content than they would for trial values of hidden problems.



Ask for the following values
Beaubourg
Carry out the following calculations
Beaubourg × 2 gives Vervel
Beaubourg + Vervel gives Population of two towns
Beaubourg + 87 654 gives Montclerc
Montclerc + Population of two towns gives Total population
Display the following results
Total population

Figure 6. Program entered by user for first problem of Figure 5.



INPUTS	OUTPUTS
Beaubourg	Total Population
125 000	587 654
120 000	567 654
122 000	575 654
120 240	568 614
121 000	571 654
120 010	567 694
120 100	568 054
120 090	568 014
120 020	567 734
120 030	567 774
120 040	567 814
120 050	567 854
120 060	567 894
120 055	567 874
120 056	567 878
120 057	567 882
120 058	567 886
120 059	567 890

Figure 7. Tabular display of values provided by user for problem of Figure 6. Target value for the variable Total Population is 567 890.

Data Sources

Data included dribble files of all the children's work at the computer, their pencil-and-paper work, observers' notes, and video recordings of every session.

Results

The findings presented in this paper are those related to the numerical approaches used by our two case-study children when solving problems in the computer environment. Where relevant, these approaches are compared with those used in their pencil-and-paper work. Results dealing with the initial phase of translating word problems into algorithmic representations are not treated in this paper.

Effect of Context

When the children began to solve each word problem on paper, doing all their own calculations, their real-world knowledge of the semantics of the problem suggested (a) suitable starting values, (b) the domain of variation, and (c) whether or not the situation was an increasing or decreasing one. For example, with Problem 3 of Figure 5, they remarked that, as the number of years increased, the remaining population decreased. Similarly, for Problem 2 they stated that, as the number of

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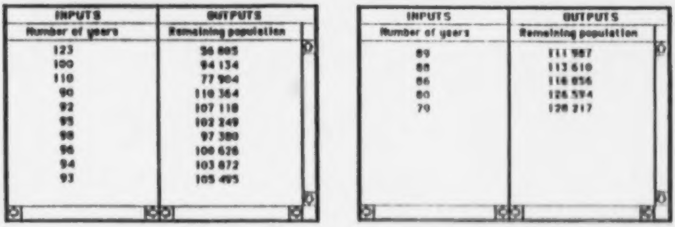
teachers increased, so did the total enrollment. They also suitably used decimals as trial values if the desired result had a decimal and used even numbers if, for example, the trial values were to be divided by two and the target was a whole number. These various context-effects were, however, not seen when the children worked with tabular displays.

As soon as they moved from their pencil-and-paper calculations to the tabular display, they tended to forget the contextual information they had previously been relying upon in their non- computer work and used strategies based solely on order relations and pattern recognition. Figure 8 shows the trial values used by one of the children for the third problem of Figure 5 in his attempt to reach the target of 128 217. That the children did not, in general, use contextual cues in deriving their guess-and-test trial values while in the tabular mode was further evidenced with the "hidden problems." The children used the same approaches to find the target output of the hidden problems as they used for the word problems.



INPUTS	OUTPUTS
Number of years	Remaining population
123	56 805
100	94 134
110	77 904
90	110 364
92	107 118
95	102 249
98	97 380
96	100 626
94	103 872
93	105 495

INPUTS	OUTPUTS
Number of years	Remaining population
89	111 987
88	113 610
86	116 856
80	126 594
79	128 217

Figure 8. Trial values used to reach the target output of 128 217 (third problem of Figure 5).

Types of Approaches Used

Two main strategies were used to derive test values in tabular form: Increasing and Decreasing. These were the principal strategies seen, primarily because the majority of problem situations presented to the children were locally either monotone-increasing or monotone-decreasing.

The Increasing strategy, the one most commonly used, can be represented by the following rules:

If the result is too small (with respect to the target),
then increase the trial value.
If the result is too large (with respect to the target),
then decrease the trial value.

The initial use of the increasing strategy in most problems was generally marked by a search involving fairly gross test values until a result was found that had the same number of digits and the same first digit as the target. The search then gradually became more refined--in the sense that smaller increments were used--based on such substrategies as interval determination, bisection, comparison of variation, asymmetry, digit-by-digit, additivity, and partial additivity. Although only the latter three of these substrategies were erroneous ones, the use of the various increasing substrategies was, in general, not very efficient, with certain test values tried more than once and several occasions of attempting to narrow-in on the wrong interval. The following excerpts illustrate the use of the digit-by-digit and the partial additivity substrategies.

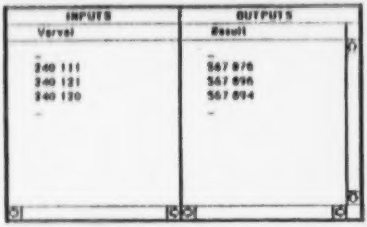
In the digit-by-digit example (see Figure 9), the subject decided that, since the first four digits of the output (567876) matched the first four digits of the target (567890), the first four digits of the input value (240111) were correct. Consequently, she attempted to increase the fifth digit of output from 7 to 9 by increasing the fifth digit of input from 1 to 2. She became extremely frustrated when she wanted to decrease the last digit of output from 4 to 0 (last line of the table shown in Figure 9) by decreasing the last digit of input, but could not--it was already at zero. She even asked, "Can I put minus 1 in place of the zero?"

In the partial additivity example (see Figure 10), we see the use of a substrategy that resembles the digit-by-digit substrategy, except that, in this case, the increment to the digit of the input value is the same as the increment required to have that digit of the output match that digit of the target. For example, in the first line of the table shown in Figure 10, the second digit of the output (85316) is 1 less than the corresponding digit of the target (86174). So 1 was added to the corresponding digit of the trial input value (7756 to 7856). The resulting output then had two digits that were correct with respect to the target. The same strategy was then applied to the third digit, and so on, until the subject realized that the strategy was no longer working.

The decreasing strategy can be represented by the following rules:

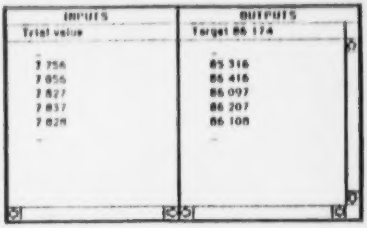
If the result is too small (with respect to the target),
then decrease the trial value.
If the result is too large (with respect to the target),
then increase the trial value.

Attempts to use this strategy generally showed confusion and interference from the preferred increasing strategy. A good example of this is indicated by the protocol illustrated in Figure 8. When the subject saw the result of his trial value 90 (line 4 of the tabular display), he said, "Still too large--no, it's not large enough." This suggests that his attention shifted from decreasing the trial value to increasing the output and subsequently using the increasing strategy on his next trial input. It was only three trials later that he realized with a certain amount of shock that he was getting farther from his target goal: "Why is it that when I put more, I get less?" The very strong tendency of our subjects to modify the input in the same direction that they desired for the output led them several times into frustrating dead ends for those problems where the input had to be modified in the opposite direction.



INPUTS	OUTPUTS
Vervel	Result
–	–
240 111	567 878
240 121	567 898
240 120	567 894
–	–

Figure 9. Excerpt of protocol illustrating use of digit-by-digit substrategy (Problem 1 of Figure 5 – target 567 890).



INPUTS	OUTPUTS
Trial value	Target 86 174
–	–
7 756	85 316
7 856	86 416
7 827	86 097
7 837	86 207
7 828	86 108
–	–

Figure 10. Excerpt of protocol illustrating use of partial additivity substrategy (Hidden Problem -- target 86 174).

Choice of trial values

The subjects were very reluctant to choose truly random guesses for their trial values. We noticed that, for their first guess, they often used the solution of the last or next-to-last problem; or, in some other cases, they divided the total amount by the number of objects, knowing very well that this would not produce automatically the desired result. We can also view some of the above mentioned substrategies in this context: The choice of their next trial value had to be "calculated" from previous tries. The idea of using any trial value just to see what it yields seems somewhat incompatible with (if not barred by) the current "recipe" approach in school mathematics.

Impact of the chosen input variable(s)

When trying to solve word problems by trial-and-error, one has to isolate one (or several) initial variable(s), which will in turn take on many trial values. The input variable chosen can have a great impact on the process of finding a solution. For example, take the problem that begins: "Quebec City has 9 times the population of Rimouski ... " It seems natural to choose "the population of Rimouski" as the initial variable; knowing the value of this variable enables us to calculate "the population of Quebec City," and go on with the problem. However, we noticed, in some cases, the tendency of subjects to choose as the initial variable the variable occurring first in the problem statement. For our example, this would be "the population of Quebec City," from which "the population of Rimouski" could then be calculated by dividing by 9.

Although this second choice is equivalent to the first one from a mathematical point of view, it can make the search for a solution much more difficult because it introduces both a wider range of integer values to be tried and the possibility of decimal numbers as results.

Conclusions

The numerical solving approaches used by the pre-algebra students observed in this study suggest that several factors tended to thwart their attempts to be systematic in deriving test values: (a) decreasing situations (i.e., situations where the trial values had to be decreased in order to increase the result, and vice versa), (b) numbers with more than five digits, and (c) numbers involving decimals. The strategies they used, though primitive, and often inefficient, eventually yielded the targetted result. The fact that the computer handled the laborious calculations allowed them to focus on certain patterns in the numbers. Some strategies we might have expected to see, but did not, were: proportionality, interpolation, and reliance upon the given relations of the word problem.

That there were no context-effects while our subjects were generating trial values in the tabular mode was a surprising finding. It seemed that the tabular representation was very conducive to non-contextual processes based simply on order relations and number patterns, and suggests that we might expect similar behaviour with other tabular forms of data, such as spreadsheets.

Since the major aim of the study was to uncover students' naturally-occurring approaches to guess-and-test methods in a computer environment, no attempt was made to develop suitable teaching methods aimed at improving students' ability with such methods. Though the number of children observed was very limited, we have at least made a start towards understanding students' primitive approaches to guess-and-test methods. We are now in a better position to design instruction that takes their approaches into account rather than ignoring them.

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TEACHING EQUATION SOLVING: A TEACHING EXPERIMENT WITH A CONCRETE SYNTACTIC APPROACH

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Abstract

This paper reports on a teaching experiment on the linear equation solving. Such experiment was carried out with two groups of 12-14 year olds in a school centre of Mexico City and a concrete-syntactic approach was applied. The results are described in terms of the modifications observed with regards to: a) the pre-algebraic knowledge and skills and b) the spontaneous (intuitive) solving methods displayed by the students before the instruction phase took place.

The results reported provide basis to elaborate hypothesis about modification of pre-algebraic profiles, when children start to learn school algebraic methods. In this way, elements to build up theoretical hypothesis, concerning the acquisition of algebraic language, are provided and suggestions at a teaching level are made.

Antecedents and Goals.- At the beginning of the acquisition of algebraic language, the learning equation solving constitutes an interaction field between the arithmetic and the algebraic thought. Deep transformations of fundamental arithmetic concepts and skills are required to overcome the difficulties which are present in such field.

According to Kieran [3] the equality sign is interpreted as an operator within the arithmetic terrain, that is, the right hand side of the equation represents *the result* of performing the operations indicated in the left hand side. For instance:

(1) $20 + 13 = 33$ or, in the pre-algebra:

(2) $x + 17 = 95$ where 95 is *the result* of adding 17 to an unknown quantity (x). Solving equations like (2) only requires to operate on the *data* (95 and 17) and such an arithmetic interpretation of "=" is sufficient to cope with them. For that reason, linear equations of the form $Ax \pm B = C$, with A,B and C, particular given numbers, are named *"arithmetic equations"*. In contrast, equations of the form $Ax \pm B = Cx \pm D$ are named *"non-arithmetic"*, as an algebraic interpretation of "=" is necessary to approach them. That is, in these equations,the equality sign is not any more an operator and becomes an equivalence sign, which links two expressions of the same algebraic nature (both expressions involve the unknown quantity) [4]. Thus, equation solving is a task which involves algebraic conceptualization of former arithmetic objects.

On the other hand, the clinical study *"Operation of the Unknown"*, carried out at the Centro de Estudios Avanzados and a school centre in Mexico City, revealed that not only *the interpretations* (meanings) of the symbolic objects involved in the algebraic tasks pressupose important conceptual changes, when such objects are transferred from the arithmetic to the algebraic terrain, but that also *the senses of actions* performed with such objects pressupose processes of transformation and (or) extension,at this transition stage. For instance, the *"extension"* of the elementary arithmetic operations (usually perform-

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ed with numbers) to the unknown quantities appears as a hard step to make, since a strong anchoring to former arithmetic actions is present for a long time. Very often, this fact manifests itself as *"the erroneous operation of the unknown"* (12x + 4 = 7x + 9 → (12 + 4 - 7...)) [illegible]. Thus, strategies for equation solving involve *meanings* of the symbolic objects and *senses* of the actions performed with them; therefore, it is a rich field for to study of interaction between two ways of thinking (the arithmetic and the algebraic) and the way that the assimilation of school methods affect the *meanings* and *senses* already installed in the pre-algebraic thought of the students. The experiment described in this paper takes this view as starting point and focusses on the elaboration of hypothesis about the short and long term effects of the school treatment.

Teaching approach.- Two kinds of teaching models were used in the teaching experiment:

1. Semantic models, characterized by providing extra-algebraic meanings for the objects and operations involved in the equation solving. These models were used in two directions: a) Taking a *"concrete"* situation as starting point, an equation is stated at a syntactic level. b) Given an equation, written in algebraic syntax, their elements are translated into a *"concrete"* situation, to solve it within this context. Thus, the actions are carried out in the *"concrete"* model and an algebraic syntax is intended to be filtered from them.
2. Syntactic models, with the aim to install the algebraic syn

tax previously developed within a *"concrete"* context. Two syntactic models were used: a) the eulerian model (or the *"doing the same thing on both sides"* model) and b) the vietian model (or the *"transposing terms"* model).

Note: Some of these models (both, semantic and syntactic) were tried in the study *"Operation of the Unknown"* already mentioned, where clinical interviews with a teaching phase were analysed. The correspondent results are communicated in severarl documents [5, 6, 7].

The population of the study.- Two groups of 12-14 year olds of the school Centre "Hermanos Revueltas" in Mexico City were chosen to undertake the experiment. Children in both groups (33 altogether) displayed different pre-algebraic skills levels (low, medium and upper levels) detected through a pre-algebra test at the beginning of the teaching stage. At the moment of the experiment, the students had learnt school methods to solve *"arithmetic equations"* (inverting operations on the numerical instances of the equation) but they had not been taught how to solve the *"non-arithmetic"* ones.

Steps of the teaching experiment [2].-

1. Administration of two pre-tests:
    a) A pre-algebraic test to measure proficiency in solving *"arithmetic"* equations.
    b) A pre-test to detect spontaneous (intuitive) strategies to solve *"non-arithmetic"* equations.

II. Teaching phase.- Two different sequences of solving models were tried with two different groups of children:

a) A concrete model *to solve* "arithmetic equations.

b) A concrete model *to state* "non-arithmetic" equations.

c) A concrete model *to solve* "non-arithmetic" equations:
GROUP I: The Balance Model ; GROUP II: A Geometric Model

d) A syntactic model *to solve* "arithmetic" and "non-arithmetic" equations:

GROUP I: The Eulerian Model (or the "doing the same thing on both sides" model).

GROUP II: The Vietian Model (or the "transposing terms" model).

III. Administration of an intermediate test (Ia) to detect the possible modifications of the old methods to solve *"arithmetic"* equations.

IV. Administration of two post-tests: Tests Ia) and Ib) were used as immediate post-tests and as differed post-tests (two months after the teaching phase took place).

Results.- In this article, only the short term effects of the experiment are reported:

1.- Modification of the Pre-algebraic knowledge

In most of the cases, the pre-algebraic methods to solve *"arithmetic"* equations were replaced by the use of one model which applied to both, *"arithmetic"* and *"non-arithmetic"* equations. This alteration does not appear in a uniform way in the population. Even more, not all the students became more efficient in solving equations. Some of them presented a loss of their previous skills to solve the *"arithmetic"* equations and made mistakes when they tried to apply the new methods, while others showed a return to their spontaneous and intuitive methods.

2.- Modification of the intuitive strategies

In general terms, the experimental population achieved a proficient level in solving equations. Nevertheless, the strategies used do not appear in a uniform way. It was possible to detect and classify 27 different previous strategies and the *"trace"* of 10 out of them is reported in this paper. It is noticeable that most of the intuitive methods tended to disappear at the end of the teaching phase (with the exception of the trial and error strategy). A preference for the syntactic models was observed in most of the students as well as a predominace of the vietian model. The vietian model spontaneously appeared in Group I; this fact can be explained as follows: this model is an abbreviated version of the eulerian one, e.g. transposition of terms comes from intermediate steps in the eulerian model (that is, applying inverse operation steps). On the other hand, one of the most relevant results in the clinical study, mentioned above, was that children tended to abbreviate the actions that they repeatedly carried out in the model (whichever it was). This can lead us to conjecture that vietian model appears in the students, taught with the eulerian one, as a *"natural"* abbreviation of the latest.

NOTE: In the Table on the following page, when the percentage increases or diminishes, but differs from 0, children displaying that strategy in the pre-test are included.

TRACE OF SOME PRE-ALGEBRAIC STRATEGIES [illegible]

STRATEGY	PRE-TEST %	IMMEDIATE POST-TEST %
1. Term to term equalization.	10.8	0 (Substituted by a verification method).
2. "Closing" the right hand side. ($x + 9 = 9 + 3 \rightarrow x + 9 = 12$)	3.2	5.8
3. Asigning different "arbitrary" numerical values to different occurrences of the unknown.	10.4	0.8 (Substituted by a syntactic model).
4. Inverting operations.	5.2	13.2 (Extension to the syntactic models)
5. Trial and error.	8.4	7.6 (Substituted by the vietian model)
6. Interpreting Ax as a place value notation.	0.4	0 (Rectified with a verification method).
7. It is already solved (In $x + 3 = 5 + 3$)	5.2	0

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DIAGRAMS AS AN INTERMEDIATE STEP IN THE DEVELOPMENT OF ALGEBRAIC REPRESENTATION

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This is a theoretical paper that describes an instructional approach to the development of algebraic representation based on a constructivist epistemology. Data is presented which emphasizes the extent to which even advanced high school students do not develop sufficient understanding in this area. Algebraic representation is analyzed in terms of two component abilities. Diagram drawing is proposed as an intermediate step which allows students to develop these component abilities one at a time in a sequence that allows for continual building on prior understandings.

The inability of students, who have taken high school algebra, to represent word problems with algebraic equations has been documented in a number of studies (Clement, Lochhead, & Monk 1981; Carpenter et al. 1983). This inability is consistent with and related to students' difficulties with all but the simplest arithmetic word problems (Carpenter et al. 1983) including problems which are multi-step, non-routine, or which contain extraneous information. This paper will further document students' lack of success with algebraic representation with some recent data on high school students. This will be followed by a theoretical development of an instructional approach to this pedagogical problem. This approach is based on a constructivist view of learning.

DATA

The problem below was given to students who were in a course beyond Algebra I. Only calculus students were not included. The problem was given at an upper-class suburban high school, a rural regional high school, and an urban private girls' high school. Data from the suburban high school appears in the table below. It is representative of the data from all three schools.

> Write an algebraic equation or equations that would allow you to solve the following problem. Do not solve.
>
> Barb is 8 years old. Her mother is 31 years old. Their birthdays are both on September 29th. When will Barb's mother be three times as old as Barb?

This problem was chosen for two reasons. First, the problem deals with a real world context, age, with which high school students are familiar. Interviews with students prior to the large-scale, paper-and-pencil task had indicated that students understood what the problem was asking and could generate informal solution strategies. The problem was also selected because it does not lend itself to word-by-word translation.

The data (See Figure 1.) reveals that difficulties with algebraic representation are widespread, continue through Algebra II and Precalculus, and that even the honors students struggle with such tasks.

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	Geometry	Algebra II	Precalculus Algebra III	Totals
Low	$\frac{0}{46}$ = 0%	$\frac{1}{33}$ = 3%	X	$\frac{1}{79}$ 1%
Regular	$\frac{9}{83}$ = 11%	$\frac{9}{56}$ = 16%	$\frac{6}{23}$ = 26%	$\frac{24}{162}$ = 15%
Honors	$\frac{9}{38}$ = 24%	$\frac{16}{42}$ = 38%	$\frac{28}{55}$ = 51%	$\frac{53}{135}$ = 39%
Totals	$\frac{18}{167}$ = 11%	$\frac{26}{131}$ = 20%	$\frac{34}{78}$ = 44%	$\frac{78}{376}$ = 21%

fig. 1

ANALYZING THE TASK

It is useful in analyzing algebraic representation to see it as composed of two abilities: (1) the ability to understand the problem (understanding the problem often depends on representing the relevant mathematical relationships using a meaningful (to the solver) symbol system (Hayes, 1981; Mason, 1984).) and (2) the ability to represent mathematical relationships using algebraic equations. The latter ability involves an understanding of variable and equation.

The possibility exists that students are missing either one or both of these component abilities. It is those students who are lacking in both that provide the greatest pedagogical challenge. Such students are being asked to represent relationships that they cannot identify using a symbol system which is without meaning for them. Such demands are overwhelming. As a result, teachers often resort to rote procedures in order to smooth over these profound difficulties.

THE USE OF DIAGRAMS

The subdivision of algebraic representation into two components emphasized two points. One is that representing a problem is often a necessary part of understanding it. The second is that since algebraic representation is often not a meaningful representation system to the student, it is not a useful tool in understanding the problem.

Diagrams could aid in the understanding of the problem. The use of diagrams, spatial representations which make use of area or length to represent quantitative relationships, is proposed as an intermediate step in understanding algebraic representation. This intermediate step could provide the meaningful representation that students need to understand the problem and provide a concrete foundation for the concepts of variable and equation.

The use , however, of a concrete referent is not sufficient to assure learning of the desired abstract system, in this case algebraic representation. Booth (1987) summarizes results of his work and that of Hart:

This lack of success in helping children develop referentially meaningful symbolic procedures has been suggested to derive from an inadequate relating of concrete and symbolic representations (e.g. Hart, 1987). This in turn may derive from:

(a) insufficient attention to the kinds of methods or models which children may have available prior to instruction (Booth, 1981);
(b) The use of concrete situations which do not appropriately mirror the formal model or procedure (Booth, 1976; Hart, 1986); and
(c) inattention prior to or during teaching to the prerequisite concepts or skills upon which the procedure being taught depends (cf Hart, 1986).

These three issues must be addressed in the development of an instructional approach that is based on a concrete referent. Following, they are used to examine the appropriateness of a diagram approach.

a) Children's Models: Simon (1985) showed that students possess at least the rudiments of a diagram system of representation which often allows them to take a "conceptual" rather than a "procedural" approach to word problems. When comparing their diagram solutions to their algorithmic solutions, students often expressed greater confidence in the diagram solution since they could "see it."

Students seem to have a sense about the relationship between various lengths or areas which they do not necessarily have about X and 2x+1. Consider the following problem and its diagram solution.

> A class was 3/5 girls. If the number of boys were doubled, and 6 girls were added, there would then be an equal number of boys and girls. How many students were in the class at the outset?

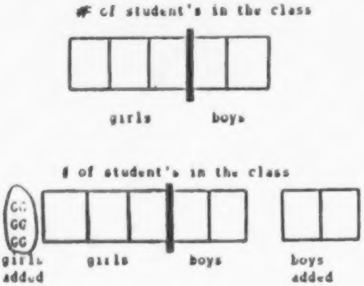
The diagram approach to this problem is accessible to students who have had no instruction in algebra (See figure 2.). Diagram representations of this type are, nonetheless, dependent on certain prerequisites which will be discussed in consideration of Booth's third point below.

b) Concrete Mirrors the Formal: Booth's second point is particularly important when viewing learning as a process of construction. If the abstract work (the algebraic representation) does not follow from the concrete referent (the diagrams), then students will be unable to construct an understanding of the abstract system based on their previous representation system. It is essential that the symbol system provide a representation for generalizing and abstracting the work that has been done at the more concrete level.

Diagram solutions generally have parallel algebraic solutions. This allows for a "connecting level" of instruction (Burns, 1979) which creates the bridge from the diagram approach to the formal algebra. Such a bridge is appropriate only after the students have become facile with diagram solutions.

The problem used above gives an example of the correspondence between the algebra and the diagrams (See figure 2.).

c) **Prerequisite Concepts:** Diagram solutions to algebraic problems, like algebraic solutions, require prerequisite understandings of whole number operations and fraction concepts. The example above calls into play abilities in representing additive and multiplicative relations and simple fractions and the concept of fractions being based on equal parts. By developing diagram representation beginning with whole number problems, then fractions, and finally algebraic problems, two categories of prerequisites are developed. First, students develop a more complete understanding of multiplication, division, and fractions and gain confidence and competence in solving word problems. Second, students' diagram representation skills become more sophisticated (Simon 1986).



x = number of students in the class

$\frac{3}{5}x$ = number of girls

$\frac{2}{5}x$ = number of boys

$$6 + \frac{3}{5}x = 2\left(\frac{2}{5}x\right)$$

fig. 2

THE INSTRUCTIONAL MODEL

The following is an instructional sequence which incorporates the ideas developed above:

1. Teacher assesses student understanding of whole number operations and fraction concepts and operations.
2. Students work with diagrams to develop concepts which were found to be weak in #1.
3. Students develop diagram solutions to algebra word problems.
4. Teacher introduces algebra symbols to represent components of diagrams.
5. Students solve algebra word problems using diagrams and then translate the diagrams into equations.

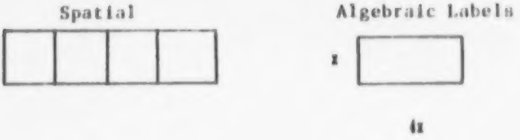


fig. 3

6. Students label diagrams with algebraic symbols which reduces the amount of spatial detail required to represent the problem. (See figure 3.) [Step 5 or 6 might not be necessary or the order of the two steps could be reversed.]
7. Students represent word problems algebraically using diagrams only when needed.

This is a proposed sequence for instruction. As with any instruction based on a constructivist view of learning (Mundy, Waxman, & Confrey, 1984), student learning must be monitored and instruction modified accordingly. Success on any of the steps outlined is based on competence at the preceding level.

SUMMARY

This theoretical approach to the development of algebraic representation is based on a constructivist view of learning and is consistent with the work of Booth and Hart (Booth 1987).

According to a constructivist view of learning, the development of new understandings is achieved by the learner through active involvement in new problems using previously developed understandings. A large percentage of students have not successfully constructed understandings of algebraic representation. A constructivist view of these difficulties would suggest that instruction did not progress in a manner which permitted these students to build on their previous understandings. As a result the students and their teacher resorted to rote memorization and imitation to make up for inadequate understandings.

The approach outlined above includes an intermediate diagram step which allows the separation and sequencing of two important components of algebraic representation, understanding and representing mathematical relations and the translation of mathematical relations into algebraic symbols. It is proposed that this separation and sequencing will allow more students to construct necessary understandings.

Only minimal anecdotal evidence has been collected to support the approach. Research studies need to be developed to investigate the usefulness of such an approach in students development of algebraic representation. It will not be sufficient, however, to investigate the effect of diagram drawing in algebra classes. The experimental group must have teachers who can facilitate and monitor students' construction of understanding. This is not a teacher-proof approach. It is dependent upon teacher decision-making (with respect to sequencing and choice of problems, pacing, etc.) which is sensitive to the learning of the students involved.

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Quantitative Concepts as a Foundation for Algebraic Reasoning: Sufficiency, Necessity, and Cognitive Obstacles[1,2]

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This premise was investigated: Quantitative reasoning provides a genetic foundation for algebraic reasoning in applied situations. A model-as-cognitive-objective was developed and tested with 25 average-ability eighth graders over a 6-month period. The premise received indirect confirmation, in that students who were severely deficient in quantitative reasoning as characterized by the model were also poor at arithmetic and algebraic problem solving.

A primary concern in mathematics education, at least within the United States, is to incorporate aspects of algebraic reasoning into the elementary curriculum (NCTM, 1987; Kieran & Wagner, 1988). That is to say, students' study of arithmetic is supposed to prepare them for algebra and currently is not doing so.

How transitions between the study of arithmetic and the study of algebra might be made is an important area currently being researched by many in the PME (see Kieran, 1988, for an excellent review). Difficulties caused by that transition is another important research area (Booth, 1981; Küchemann, 1980; Herscovics & Kieran, 1980; Vergnaud & Cortes, 1886).

The premise of this research is: *Competence in algebra is founded genetically upon competent quantitative reasoning*. This premise is similar to a view of algebra as "generalized arithmetic," but is different in key aspects, for here it is also argued that *arithmetic* is founded on quantitative reasoning.

The claim that quantitative reasoning is genetically foundational to formal algebra

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[1] The research reported in this paper was supported by the National Science Foundation grant MDR 87-51381 and by Apple Computer, Inc. Any conclusions or opinions expressed are those of the author and do not necessarily represent the positions of NSF or Apple Computer.

[2] Software reported in this paper and associated materials can be obtained from the author by sending a 3.5" double-sided, double-density disk and a cheque for $5.00

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does not preclude the importance of "non quantitative" algebra (e g , the study of group, ring, and field properties of polynomials, or the study of algebras of real- and complex-valued functions). Rather, it provides a link between "non quantitative" algebra and the algebra of applications.[3]

QUANTITATIVE REASONING

A quantity is a measurable quality of something. A magnitude of a quantity is the quantity's measure in some unit. Quantities are categorized into two kinds: extensive and intensive quantities (Freudenthal, 1973; Schwartz, 1987). An extensive quantity is one which may be measured in absolute units. An intensive quantity is one which may be measured only by units that are constructed comparatively. Lengths, areas, volumes, cardinalities, and ordinalities are examples of extensive quantities. Speeds, densities, temperatures, pressures, and pitches are examples of intensive quantities, as is a multiplicative comparison (ratio) of two quantities.

To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities. Quantitative mental operations include these:

Operation	Example
• Combine quantities additively	*Unite two sets; consider two regions as one.*
• Compare quantities additively	*"How much more of this is there than that?"*
• Combine quantities multiplicatively	*Combine distance and force to get torque; combine linear dimensions to get regions*
• Compare quantities multiplicatively	*"How many times bigger is this than that?"*

Quantitative reasoning is to reason about situations in terms of quantities and quantitative operations. This characterization does not identify quantitative reasoning with arithmetic; in fact, it is independent of arithmetic. A quantity within a structure of quantitative operations may require any arithmetic operation to determine its value.

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[3] I have maintained elsewhere that even the more structural features of elementary algebra can be taught with an explicit foundation in arithmetic. See Thompson and Thompson, 1987.

The appropriate operation to compute a quantity's value can be inferred only within the context of a quantitative structure together with assumptions about what values are known.

Complex quantitative reasoning entails relating groups of quantitative mental operations, such as in forming a multiplicative comparison of an additive comparison and an additive combination (i.e., "How many times bigger is this difference than is this combination?"). Quantitative reasoning also entails reasoning relationally about quantitative structures, entails the constitutive mental operations for comprehending a quantity situationally, and entails the constitutive mental operations which allow one to recognize a quantity as one whose value varies or can vary.

To investigate the sufficiency of quantitative reasoning as a foundation for arithmetic and algebraic reasoning, the author constructed a computer program (called *Word Problem Assistant*, or *WPA*)which makes numerical or algebraic inferences solely on the basis of quantitative structure (Thompson, 1988). Once given a representation of a situation in terms of quantities, units of measurement, initial numerical information, and pairwise relationships among quantities, the program will generate arithmetic or algebraic expressions for computing unknown values within the situation, including the setting up of equations where appropriate. For example, here is a situation with a question about it.

> MEA Export is bidding to supply an oil valve to Costa Rica. They sell this valve for $5000. Freight charges to Costa Rica are $100. The insurance cost is 1.25% of the total price to Costa Rica. The total price to Costa Rica includes the cost of the valve, the cost of freight, and the cost of insurance.
>
> What is the total price to Costa Rica?

Figure 1 shows a representation of this situation made within WPA, and also shows the inferences WPA made. The boxes in Figure 1 (called notecards) represent

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quantities.[4] The bottom cell in a notecard contains the name of the unit in which the quantity is measured. The *Value Cell* in a notecard holds the quantity's measure, if one is known. The *Description Cell* of a notecard is a place that holds an expression which tells how to compute the quantity's value.

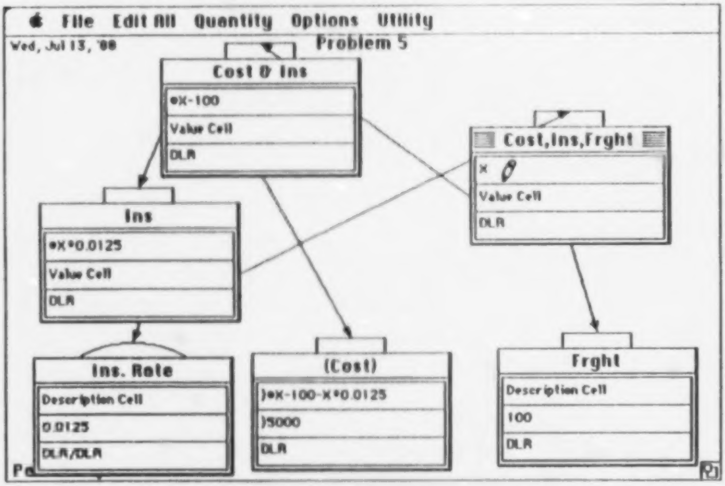
File
Edit All
Quantity
Options
Utility
Wed, Jul 13, '88
Problem 5
Cost & Ins
•X-100
Value Cell
DLR
Cost,Ins,Frght
x
Value Cell
DLR
Ins
•X*0.0125
Value Cell
DLR
Ins. Rate
Description Cell
0.0125
DLR/DLR
(Cost)
)•X-100-X*0.0125
)5000
DLR
Frght
Description Cell
100
DLR
Pa

Figure 1. A representation in WPA of an algebra story problem, along with the inferences it made based on the quantitative information presented about the situation.

Entries with bullets (•) are the results of inferences made by WPA. Any entry without a bullet was entered by hand. The arrows show quantitative relationships. For example, the arrows coming out of **Cost,Ins,Frght** show that it is made of **Cost & Ins** together with **Frght**. Since the units are all in dollars (DLR), WPA inferred that this is an additive relationship, and hence that if **Cost,Ins,Frght** is *x*, and if **Frght** is 100, then **Cost & Ins** must be *x*-100. Other inferences were made accordingly, resulting in the inference that *x*-100-*x**0.0125 is a formula for computing the value of **Cost**. Since **Cost** already had a known value of 100, WPA decided to ignore **Cost**'s value (and indicated as much by inserting "}" before its value) and to propagate the formula—resulting in an equation (i.e., two descriptions of a single quantity's value) whose solution will determine numerical values of all quantities in the representation.

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[4] The shape of the notecards and the convention of using arrows to show quantitative relationship was developed by Valerie Shalin (Greeno et al, 1985; Shalin, 1988).

AN EXPLORATORY STUDY OF QUANTITATIVE REASONING

WPA constitutes a cognitive objective—a model which can be taken as depicting knowledge students are to build as a result of instruction (Greeno, 1980). As a cognitive objective, it can be used as a guide for instruction and as a guide for assessing students' knowledge.

WPA was used on an ongoing basis for 6 months with 25 average-ability eighth graders at the ISU laboratory school. Students were given pretests on problem structure (similar to Silver, 1981), rate and ratio, concepts of expression and variable (similar to Küchemann, 1980), and arithmetic computation. Only on the computation pretest did the class average exceed 50%. Other pretest averages were 14% (Structure), 30% (Expressions and Variable), and 41% (Rate and Ratio).

The classroom was equipped with 11 Macintosh Plus computers. Students worked in groups of two or three, whether in activities away from the computers or in problem solving at the computers. Whole-class instruction was done using the blackboard, overhead transparencies, or with a Macintosh connected to a projector. When students worked in groups, one group was selected for videotaping. If the videotaped group used a computer, their computer was connected to a projector so that both they and their computer screen could be videotaped simultaneously.

The user-interface to WPA was made so that, to use it in solving word problems, students needed to make decisions only about quantities involved in the described situation. Decisions to be made about identified quantities were: a quantity's name, its type ("number of things," difference, ratio, or rate), the unit in which it is measured, and quantities that it is made of (by some quantitative operation, but they did not have to specify the operation—WPA inferred one). The following are mistakes that students made on such a frequent basis that they are indicative of cognitive obstacles to quantitative reasoning, and hence are perchance indicative of obstacles to algebraic reasoning.

Naming Quantities

A predominant tendency was for students to name quantities by their numerical value, e.g. "250 dollars" instead of "Sam's Savings." Their failure to distinguish between a quantity and its measure hindered their ability to explicate relationships. This was for two reasons. First, students could identify only "quantities" that had given numerical values. It was difficult to identify quantities in a situation that had an unknown value, and especially difficult to recognize the need for including "intermediate," but unmentioned quantities. Second, when students had only numerical values in mind they had no rational basis for deciding what operation to perform to find an unknown value. When one has in mind, for example, only "17.32 dollars" and "12.87 dollars" without relational information about the quantities having these values, one cannot decide, on this information alone, what operation would be appropriate to compute the value of a third quantity.

Identifying Kinds of Quantities/Assigning Units of Measure

Students reliably identified and assigned appropriate units only to extensive quantities of a linear dimension. Multiplicative quantities of any sort (products, ratios, rates) were commonly mis-identified or given an inappropriate unit.

The most common mis-identification was of rates. The typical mistake was to give the "numerator" of a rate or ratio as the unit of the identified quantity (with the exception of speed in miles per hour). Pay rates per unit time were given "dollar" as the unit. Prices were given "dollar" or "cent" as the unit. Ratios between children and buses were given "child" as the unit. And so on.

WPA required students to pick the kind of quantity a notecard was to represent and to supply the unit in which the quantity was to be measured. When a mismatch occurred (e.g., saying that a rate had "meter" as a unit), it would say that the unit and kind of quantity were not consistent, and would not let students proceed until the two were made consistent.

Showing Relationships

WPA checked compatibility of units and kinds of quantities when students attempted to show relationships among quantities. If they attempted to show, by drawing arrows, that an extensive quantity measured in "meter" was made by combining a rate measured in "ft/sec" and a time measured in "sec," WPA would explain that the quantities could not be related in this way, and would not establish a relationship (i.e., would not draw the arrows). The constraint of compatibility was one which students bumped against commonly.

Students' consistent violation of the two constraints imposed by WPA proved to be so severe that direct instruction on quantity, units of measure, and relationship was necessary. This instruction, however, uncovered affective and sociological norms which severely obstructed students' abilities to assimilate instruction on quantitative reasoning and its generalization to algebra. Another publication will address issues of instruction, affect, and effects of the "social contract" of schooling.

CONCLUSION

The premise that quantitative reasoning provides a genetic foundation for arithmetic and algebraic reasoning remains to be tested directly. The current study provides, at best, indirect confirmation of the premise. This is that students who are not good arithmetic or algebra problem solvers have substantial deficiencies in quantitative reasoning. The next phase of this project will be a longitudinal study of quantitative reasoning and its relationship to algebra. We will follow a class of fifth-graders for four years, instructing them in quantity, quantitative reasoning, and the representation of quantitative structure. It is hoped that a longitudinal approach will illuminate issues regarding sufficiency of quantitative reasoning for algebra

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A Comparison Among Children, Adolescents, and Adult Learners On Selected Area Measurement Concepts

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The responses of ninety-five students to three area measurement problems were analyzed to compare differences in conceptual understanding among learners from different age groups, and to identify factors which influence area learning. Findings indicate age related trends when children, adolescents, and adults were asked to compare sizes of two rectangles, and age related differences when children were asked to distinguish between area and perimeter concepts in an applied problem. Partitioning concepts required to construct equivalent area units were lacking in practically all students.

The way children understand area measurement skills has been described in a number of recent papers. Anecdotal accounts of children's behavior related by Biggs (1987) prompted her to develop a carefully structured sequence of conceptually-based activities. In terms of the fundamental concepts, Hirstein, Lamb, and Osborne (1978) identified unit iteration and invariance as the specific subconcepts required for understanding area. Concerning unit iteration, Hart (1984) recognized that children employed the counting method for obtaining area far into the secondary school. Maher and Beattys (1986) observed that children used the square unit to iterate for rectangular and composite rectangular shaped regions, but failed to generalize the procedure to irregular figures when angles were greater or less than ninety degrees. In a related study, Heraud (1987) provided a variety of units and gave students the task of selecting an appropriate unit with which to tile each of various polygons, a circle and a closed curve. With each example, the researcher recognized that the children were influenced by the shape of the region, and concluded that their unit selection appeared to be based on the notion of similarity.

Results of the last two studies indicate that not only are the subconcepts of invarance and unit iteration necessary for area learning, but also some geometric understandings are prerequisite. More specifically, an understanding of angles and polygons would facilitate area learning. This notion is substantiated by work based on van Hiele research. In their research and development of learning units, Fuys, Geddes, and Tischler (1985) presented two modules in geometry prior to the introduction of area measurement concepts. The two modules which precede area dealt with classification of two dimensional shapes, and angle mesurements and relationships; the researchers determined this work, on van Hiele levels 0, 1, and

2, was needed to work successfully at levels 1 and 2 on the area module. While the van Hiele model characterizes hierarchical, developmental levels of geometric thinking, the levels are perceived as more functionally related to instruction than to age or maturation. Van Hiele (1987) addressed the issue directly, stating that he has taught geometry to some people thirty years old, and they had the same problems as children in learning it.

Based on these assertions, and findings from an ongoing study, this paper addresses three questions: Aside from unit iteration and invariance, are there other factors which influence area measurement learning? Does van Hiele's position that adult learners have the same difficulties as children apply to the learning of area measurement concepts? If there is evidence of this, what are some of the similarities and differences in how children and adult learners think about area?

Methods

Data were collected from three separate populations of urban minority students. The participants included thirty-one elementary students (9-11 year olds), thirty-one ninth and tenth graders (14-15 year olds) enrolled in a general mathematics class, and thirty-three adult community college students (20-40 year olds) registered in a remedial mathematics class. Each group of students participated in the same three problem solving tasks.

The first two activities dealt with rectangular regions. For these problems, surfaces with dimensions were given; the rectangles were not gridded. For these two activities, geoboards, plexiglass square-inch embodiments, waxy strings, rulers, unlined paper and graph paper were accessible to the students.

The first activity required students to compare the size of two scaled drawings of rectangles, one that was 7 ft. by 3 ft., the other 6 ft. by 4 ft. In the second task, students had to imagine that both of the rectangles previously described were posters. First, they had to decide which poster, if either, required more glass to cover it. They were then asked to decide which poster, if either, would require more wood to frame it. As was the case in the first problem, students were asked to explain their answers. The third task required students to consider a representative drawing of a square foot, and to draw a square yard relative to that square foot.

Sessions were audio and /or videotaped, and results were analyzed from coding sheets and student worksheets.

Results

When asked to compare rectangles, nearly four-tenths of the children, a third of the adolescents, and a fourth of the adults responded that the sizes were the same. Of these responses, two age-related trends were apparent: the older the students, the greater the number of responses based on perimeter, and the younger the students, the more responses based on compensating a greater length of one figure for a greater width of the other figure. A summary of results are shown in Table 1.

Table 1 Comparison of Rectangular Areas

Age Group	Same			Different		
	P[1]	A[2]	C[3]	A[4]	N[5]	S[6]
9-11	.06	.10	.23	.00	.00	.61
14-15	.13	.00	.19	.16	.06	.45
20-40	.18	.00	.06	.30	.09	.36

1. P: Calculated perimeter, 2. A: Added 2 sides, 3. C: Compensated length one unit longer in A, width one unit longer in B, 4. A: Computed area, 5. N: Identified numbers, 6. Identified shapes.

Results indicate that the majority of 9-11 year olds believe that the size of a region is a function of its shape. This idea is also prevalent in adolescent and adult thinking, though the proportion that respond this way decreases with age. A correct area solution was never suggested by any of the children; some adolescents (16%) responded correctly, as did almost one-third of the adults.

Results on the second task, found in Table 2, demonstrate that practically all students have trouble distinguishing between applications of perimeter and area.

Table 2 Area and Perimeter Applications

Age Group	C[1]	A[2]	P[3]	D[4]	L[5]
9-11	.00	.00	.02	.13	.85
14-15	.13	.16	.23	.23	.25
20-40	.06	.30	.33	.15	.12

1. C: Correct solution, 2. A: Area solution for both, 3. P: Perimeter solution for both, 4. D: Associated greater length with greater perimeter and greater width with greater area, 5. L: One linear dimension associated with larger area and perimeter.

Of the 9-11 year olds, 52% responded that the same amount of glass and wood would be required for each poster. Similarly, 39% of the adolescents and 63% of the adults chose either an area solution or a perimeter solution for both problems

Two different linear interpretations were suggested to compare size. One group of students decided that a particular figure needed more glass and wood because of one of its linear values (one figure was longer, or the other was wider). A second group selected the longer region as needing more wood, and the wider region as needing more glass.

Responses to the third task, illustrated in Table 3, were diverse with two exceptions.

Table 3 Unit Comparison

Age Group	N[1]	R[2]	A[3]	T[4]	S[5]	L[6]	C[7]	D[8]
9-11	.05	.10	.33	.05	.10	.38	.00	.00
14-15	.07	.07	.40	.00	.07	.20	.13	.07
20-40	.07	.07	.29	.00	.14	.29	.00	.14

1. N: Drew congruent square, 2. R: Drew 2x2 square, 3. A: Drew arbitrarily larger rectangle, 4. T: Drew 1x2 rectangle, 5. S: Drew a 2x2 square (gridded), 6. L: Drew 1x3 rectangle, 7. C: Correct response, (drew 3x3 square), 8. D: Didn't know.

Most square yards were represented by larger squares; several others were drawn to the width of the square foot, but longer in length by an unspecified amount. Approximately one-third of the students in each group drew the square yard larger than the square foot, but refused to identify how much larger it should be. The second most common response was drawing a 1x3 rectangle, with the explanation based on the linear equivalence of 3 feet equal to 1 yard.

Discussion

All measurement shares certain characteristics. Linear or angular measurement is obtained by iterating an appropriate segment. Area measurement simply extends this concept to the size of a planar region. But problems can arise with area measurement that are not encountered with either linear or angular measurement; this study identifies some of these problems.

The notion of 'size' for planar regions is ambiguous for most students; this is consistent with Wilson's observations (1986) concerning children's concepts of rectangles. Regions have different shapes as well as different sizes. But when asked to compare regions, the most common student responses indicated that size was dependent on shape. While a considerable portion (30%) of the adults seemed to recognize that area was appropriate, inconsistencies appeared in the next problem.

The second activity related applications of area and perimeter. Two response were dominant. First, similar to Hart's findings, most older students opted for working with the numbers in a typically, often incorrect, algorithmic style. More specifically, 66% of the adults and 39% of the adolescents were unable to disassociate perimeter and area, choosing either one or the other algorithmic solution for both responses. This inability to disassociate perimeter and area is formidable; extensive work by a number of researchers (Douady and Perrin Glorian,1987; Fuys, Geddes, & Tischler, 1985; Pace, in progress), supports this contention.

A second common response to area and perimeter applications was suggested more often by younger students. More children and some adolescents offered an incorrect linear interpretation for a two dimensional region. Quite often, the selection of a particular figure was based solely on its having either a longer length or a wider width.

For all students, it appears that the difficulty with unit did not begin with iteration. Partitioning, which is essential to planar measurement, was ignored. When asked to construct a model of a square yard in relation to a square foot, students were unable to do so. This inability to partition may be related to the work of Pothier (1983), who suggested the use of "appropriate number theoretic concepts in making sense of the emergence and differentiation of the partitioning process." (p.317) Some partitioning experiences based on this theory may benefit students who evidenced conflict between requirements for squaring a region, and the linear equivalence 3 ft. = 1 yd.

Unit iteration and invariance are conceptual components of area measurement. Recognizing similar conceptual processes in different types of measurement, approaching area with a background rich in geometric experiences, and applying partitioning skills based on geometric interpretations of certain number theory concepts are other components worthy of consideration as children are given opportunities to construct area concepts.

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The Development of Geometric Conceptualizations in Logo[1]

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ABSTRACT

A synthesis of Piaget's and van Hiele's theories was used to gain insight into children's geometric conceptualizations in Logo evironments and served as a psychological basis for guiding the development of instructional activities and assessments. Over two studies, the Logo children's conceptualizations of geometric objects were more generalized and mathematically-oriented. These children were more likely to view geometric objects in terms of the actions used to construct them. Thus, the Logo experience appears to have engendered within the children an explicit awareness of those action-based conceptualizations identified by Piaget as fundamental in the development of geometric ideas. In addition, it helped children make the transition from the visual to the descriptive level of thought in the van Hiele hierarchy.

CONCEPTUAL FRAMEWORK

It has been claimed that in Logo programming children learn mathematics by utilizing concepts that aid them in understanding and directing the turtle's movements (Papert, 1980). Perhaps the fundamental basis for this claim is that appropriate Logo environments can help children elaborate on and become cognizant of the mathematics implicit in certain kinds of intuitive thinking. That is, when "intuition is translated into a program it becomes more obtrusive and more accessible to reflection" (Papert, 1980, p. 145).

In the case of geometric shapes, for example, children initially identify and operate on shapes and other geometric configurations according to their appearance (i.e., they are at level 1 in the van Hiele, 1986, hierarchy). They recognize figures as visual gestalts, often referring to real-life exemplars, saying, for instance, that a given figure is a rectangle because "it looks like a door." They do not, however, attend to geometric properties characteristic of the class of figures represented. The properties are implicit in the recognized figures but are experienced by the children only intuitively. At the second (descriptive/analytic) level, children can explicitly recognize and characterize shapes by their properties. For instance, they might think of a rhombus as a figure that has four equal sides. That is, for a child at this level, the term "rhombus" refers to a collection of "properties that he has learned to call 'rhombus'" (van Hiele, p. 109).

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In Logo, such figures (i.e., geometric shapes) must be viewed as turtle "paths," that is, as traces of the turtle's movements. Children must analyze these figures in terms of such movements, thus making essential geometric components of the figures salient. In particular, because turns and straight sections (line segments) play a central role in forming figures, such analysis focuses the children's attention on these components, thereby enhancing their understanding of turns, angles, line segments, and their interrelationships. Because such action-based conceptualizations have been identified by Piaget as fundamental in the development of geometric ideas, Logo may help children become cognizant of their geometric intuitions, and appropriate Logo environments may facilitate the transition from level 1 to level 2 geometric thinking. That is, when children write a procedure to draw a shape that they recognize visually, they must analyze the visual aspects of the shape and how its component parts are put together, an activity that encourages level 2 thinking.

The goal of our research was to investigate the development of geometric conceptualizations in Logo environments. Specifically, we examined children's: (a) concepts of angle and angle measure, (b) notions of shape, form, and motion, and (c) progression through the van Hiele levels.

STUDY 1

Procedures

The subjects for the study were 48 eight-year-old children. The children were randomly assigned to one of two conditions, Logo computer programming and control. Structured interviews over such notions as angle and shape were performed at the end of the computer sessions.

The Logo children worked in pairs for three 45-55 min sessions per week for a total of 78 sessions. In the initial sessions, the FORWARD and BACK and rotation (RIGHT, LEFT) commands were introduced and dramatized. Children were given challenges such as determining the exact length and width of the screen in turtle steps, creating as many ways as they could to get to a location, and determining what input to LEFT makes the turtle turn "half-way around" (i.e., 180). Children discussed all new ideas and problem solutions and were asked to name other situations, in and out of school, in which such ideas and solutions would be useful. For instance, it was noted that a 90° turn in a path creates a right angle, so that the two sides were perpendicular. Children were led to identify this angle inside and outside of the room.

Children used a support program, TEACH, that allowed them to define a procedure and simultaneously watch it execute, editing whenever necessary. Various procedural solutions to problems were compared and discussions centered around differences in solutions (e.g., What was altered in the procedure to create this geometric effect?) Given an altered procedure (or graphic), children were asked to predict the corresponding graphic produced (or procedure that produced the graphic). Thus, there was an attempt to help children construct mappings between components of procedures and the corresponding component of the shape formed.

Fundamental geometric figures were constructed and manipulated. For example, the concept of square was discussed, with children offering definitions and exemplars. The children constructed squares using dramatization, paper-and-pencil, and Logo. They constructed several square procedures, each having a different side length. Procedures were compared, and children led to list both similarities (e.g., the pattern of commands, 90° input to the rotation commands) and differences (the input to the FORWARD commands, right or left turns). Teachers discussed geometric terms such as line segment, rotation, angle, right angle, congruence, and similarity; however, children were not requested to memorize these terms or their definitions. The children were finally challenged to plan and construct programs that used their square procedures as "building blocks" (e.g., towers, or squares inside of squares).

The concept of procedures with variable inputs was then introduced. The remaining sessions included challenges, often oriented toward seasonal interests, such as writing valentine heart procedures (and thus arcs); contests, such as writing the shortest (most elegant) program for a "stacked rectangle pyramid," or duplicating given shapes and utilizing them as many different ways as possible in the creation of a picture; explorations of regular polygons (and the "turtle total turn theorem" or "the rule of 360°"); and a final collaborative project.

The control children received computer experience under the same conditions as the experimental group (i.e., six pairs of children working with the same teachers), but the content, designed to develop creativity and

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literacy, included composition using Milliken's *Writing Workshop* (an integrated package of prewriting programs, a word processor, and postwriting programs).

Findings

The Logo group scored significantly higher than the control group on the angle section of the interview ($t(46) = 5.88$, $p < .001$) as well as on the shape section ($t(46) = 5.17$, $p < .001$). A detailed analysis of the interviews revealed several aspects of the children's geometric thinking. First, the children's conceptual scheme of angle was complex and multifaceted and appeared to include sub-schemata from the domains of common language, normal school experiences, and Logo. Second, these angle conceptualizations were not well discriminated from related concepts such as orientation, corner, rotation, "right" and so on, and often included other non-standard notions. Third, although the Logo experience had by no means eliminated "errors," it appeared to have significantly affected the children's ideas about the nature of angles.

Logo experience also appeared to engender greater awareness of the components of two-dimensional figures. Logo children were more likely to construct mathematically acceptable categories of quadrilaterals and discriminate on the basis of the properties of these figures. That is, the Logo children, more than the control children, exhibited signs of transition from van Hiele level 1 (visual) to level 2 (descriptive/analytic) thought.

STUDY 2

Procedures

Procedures were much the same as those of study 1; however, subjects were 12 fourth-graders and the interviews, similar to those in study 1, were conducted before, during, and after the treatment.

Findings

All children initially had vague notions of the concept of angle, with most conceiving of it as a tilted line segment, any line segment, or an orientation or heading (see Table 1). At the time of the interview 2, after informal experience with rotations and angles in Logo (but no classroom instruction), 2/3 of the Logo group but none of the control group altered their conceptualizations to include notions of corner. Between the interview 2 and 3, children in both groups received traditional classroom instruction on angles; the Logo group completed experimenter-assigned investigations of angles. Interview 3 indicated that all of the Logo children, compared to 1/3 of the control children, possessed a mathematically accurate concept of angle.

On all three items in which children described geometric shapes, more Logo children exhibited signs of level 2 thinking, explicitly mentioning geometric properties of shapes (e.g., Table 2). Thus, there is strong evidence from both the angle and the shape items in support of the hypothesis that Logo experiences can help children become cognizant of their mathematical intuitions and facilitate the transition from visual to descriptive/analytic geometric thinking. The fact that these children were still in transition to level 2, however, is indicated by inconsistencies between their selections and descriptions of shapes. For instance, one child distinguished between the "slant" (i.e., the nonperpendicularity of sides) of a parallelogram and the "slant" of an oblique rectangle, stating that only the latter was an acceptable slant for a rectangle. However, she then proceeded to identify all parallelograms as rectangles.

CONCLUSIONS AND IMPLICATIONS

Overall, the Logo children's conceptualizations of geometric objects were more generalized and mathematically-oriented. These children were more likely to view geometric objects in terms of the actions used to construct them. Thus, the Logo experience appears to have engendered within the children an explicit awareness of those action-based conceptualizations identified by Piaget as fundamental in the development of geometric ideas. In addition, it helped children make the transition from the visual to the descriptive level of thought in the van Hiele hierarchy. Thus, instructional activities and assessments based on a synthesis of Piaget's and van Hiele's theories can be used both to gain insight into children's geometric conceptualizations and to positively affect those conceptualizations.

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Table 1: Development of Overall Angle Conceptualization

Child	Interview 1	2	3
Control			
GZ	O/H + Seg + Inter	O/H	Inter + Union
RF	Shape	TL	Cor + Seg
BG	Inter + TL	TL	Cor + Seg
AS	TL	TL	TL
JU	TL + O/H	O/H	Union
IU	TL + Seg	TL	TL
Logo			
SC	TL	TL	Union
FF	TL	TL + Cor	Union + Turn
LG	TL	Union	Union
SD	Seg	TL + Cor + Inter	Union
KR	Seg	Inter	Union
BV	O/H	O/H	Union

Code	Conceptualization	Code	Conceptualization
Shape	A shape	Turn	Turn or rotation
Seg	Line segment or side	Cor	Corner
TL	Tilted line	Inter	Intersection of two lines
O/H	Orientation or heading	Union	Union of two lines or segments

Table 2: Number of Children Mentioning Properties Differentiating Quadrilaterals

	Control	Logo
VISUAL		
Looks like: Looks like an object (e.g., a door)	2	
Size: One is bigger (fatter, longer, etc.)	3	5
Shape: They are not the same shape.	4	2
Visual characteristic: This one has points.	2	1
Orientation: Pointing in a different direction.	1	2
Curvilinear: This one is round.	1	1
EXPLICIT PROPERTIES		
Side length: This one longer (or unequal) sides.	1	5
Parallel: This one has parallel sides.		2
Right: This one has one right angle.		2
Angle: The angles are different.		2
Convex: This one is bent in.		2
EXPLICIT CLASSIFICATION		
Shape classification: This one is a square.		2

THALES'S THEOREM , CABRI AND MAC DRAW

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ABSTRACT. The object of this research is to investigate the difficulties for the pupils when using Thales's Theorem in solving a problem. It was carried out in three environments to study how different microworlds modify pupils'strategies. There are two computer environments with a Macintosh: one with CABRI software, one with Mac Draw software. The third environment is a paper- pencil one.

I . INTRODUCTION

Thales's theorem is a key notion in mathematics and in mathematics teaching in France at the end of secondary school (14 to 15 years old). It allows a transfer from the geometrical to a numerical context where the parallelism is related to the proportionality (1985 DOUADY). The difficulties encontered are of different types. They are, however, much greater when the result required is not numerical, a case when the change of context appears necessary from the start, but a geometrical property to explain (1987 GALLOU-DUMIEL).

II . THE PROMEMATICAL ASPECTS OF THE THALES'S THEOREM

At the time of Thales right up to present time for pratical problems (the cubic volume of a growing pinetree for example) Thales's theorem makes it possible to calculate by using other lengths, a lenght wich cannot be measured directly on account of its height and technical difficulties. Thanks to Thales it is possible to change the dimension of space (1986 LABORDE). When you want to attribute a structure of vectorial space to R^2 or R^3 you use Thales's theorem to define multiplication by a scalar. Thales's theorem both historically, in its presentation in teaching and its application in problems appears to be a tool. In particular Thales's theorem enables us to transform a class of problems wich involve parallelism in a geometrical setting into a class of problems which involves proportionality in a numerical setting.

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III AIM OF THE STUDY

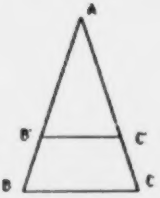
We have choosen to study the difficulties which pupils face in solving problems where Thales's theorem has to be used in the fourth year of secondary school . There are different steps in the resolution of one of the problems: decision to use Thales's theorem; choice of configurations where Thales's theorem has to be used; choice of which form of Thales's theorem to use; demonstration with calculation. We want to know if pupils in a computer environment with the two different softwares try other transformations of graphs and do different conjectures from those in a paper - pencil environment. Also we want to know if the decision to use the Thales's theorem is taken earlier and in a more spontaneously way in the computer environments. What are the difficulties that remain similar in the three environments?

IV . CHOICE OF PROBLEMS WITH STUDY OF VARIABLES

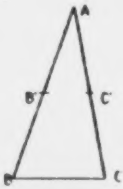
We call variable an element of a problem, a modification of whitch induces a change of strategy by the pupils. Thanks to early study (1987 GALLOU-DUMIEL) we can draw up a list of variables. The variables are the following:

type of answer: it can be a number or a geometrical caracteristic;

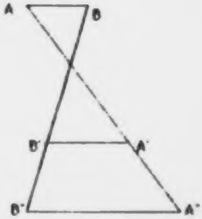
configuration with the different steps: the presence of geometrical figures: triangle, trapeze, paralelogramm ...; use of Thales's theorem in a triangle with the second form (fig1) ; use of middle theorem (fig2) ; use of Thales's Theorem with secant lines (fig3); direction of the parallels;



$$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{AB'}}{\overline{AB}}$$



B' middle of [AB]
(B'C') // (BC) then
C' middle of [AC]



$$\frac{\overline{B'B}}{\overline{B'B''}} = \frac{\overline{A'A}}{\overline{A'A''}}$$

number of all lines of the figure;

number of times Thales or its converse proposition has to be used;

presence of different forms of Thales's theorem or not;

use of algebraic measures;

algebraic computation;

change of space or not.

Six problems have been choosen with different values for variables.

V . EXPERIMENTS

problems	CABRI	MAC DRAW	PAPER-PENCIL environment
	two classes 48 pupils	two classes 42 pupils	one class 26 pupils
TRAPEZEB TRAPEZEA	24 pupils	14 pupils	10 pupils
TRIANGLEB TRIANGLEC	12 pupils	12 pupils	8 pupils
PARALLELOGRAM POPLAR TREE	12 pupils	16 pupils	8 pupils

The experiments take place after teaching of Thales's theorem in the usual class room situation . In all environments two pupils work together to get the solutions of the problems. In each class we give the pupils two texts of problems. They have to write a solution for each one. They work during one hour . In paper-pencil environment, the figures have to be drawed by the pupils. In the computer environments with Mac Draw or with CABRI, the pupils have a previously traced figure on the screen of the Macintosh computer. They have to use this figure, doing modifications with the instructions of the software. It is forbidden to draw a figure on a paper. In addition , in CABRI environment they are told the basic point of the previously traced figure and they must try deformations starting from the basic points. They have had before one hour of training for the use of the software .

VI STUDY OF THE COMPUTER ENVIRONMENTS

The first software is CABRI . It is an interactive software which permits:

1) to construct geometric figures on the screen starting from the basic points with the following instructions : segment defined by two points ; middle of a segment ; line parallel to a given line and including a given point , mediatrice of a segment ; intersections ;

2) to transform the figure starting from one of the basic points , with the conservation of all constructions realised with the instructions ;

3) to hide a line and make it appear again ;

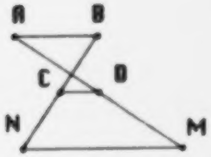
4) to ask the measure of the distance of two points .

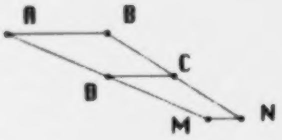
The second software is Mac Draw . The instructions are the following ones : to draw a segment between two points ; to translate a segment ; to change of thickness of a line ; to hide a segment ; to put a squared covering . The difference between the two softwares is the presence in CABRI one of basic points one of a deformation starting from a basic point .

VII . STUDY OF PUPILS PRODUCTIONS

1) Use of the deformation in CABRI software .

5 pairs of pupils on 12 ones having to solve TRAPEZEB in computer environment with CABRI have used the deformation in the following way :





When they have the second figure on the screen , they write on their paper :

$$\frac{\overline{AD}}{\overline{DM}} = \frac{\overline{BC}}{\overline{CN}} \quad \text{or} \quad \left(\frac{AD}{DM} = \frac{BC}{CN} \right) \quad \begin{array}{l} AD = DM \\ \text{then } BC = CN \end{array}$$

After, two pairs of pupils go back to the first figure and verify the rapports . The variables which have changed of value between the two figures are the variable "use of Thales's theorem with secant lines and direction of the parallels" . There has been no contestation upon the validity of the use of the second figure to solve the problem between the two pupils of each pair . Two pupils have said during the transformation "parallel lines keep parallel and D is always the middle of [AN]" . The use of the second figure is not an error . Pupils are conscious that the use of Thales's theorem is the same for the first and the second figure . Here the figure is only a support to solve the problem and not the place to proove something , there it appears the following idea: to use a property of the figure , the pupils can change it into another , similar for this property and different for other properties . The pupils working in computer environment with CABRI are those who have the best and the quickest resolutions for the problem TRAPEZEB .

2) Part of variables number of times Thales is used and number of different forms of Thales .

POPLAR TREE and PARALLELOGRAMM are the problems where the number of the uses of Thales's theorem is the most important and there is a use of converse proportion of Thales's theorem . The most successful pupils for these problems are the pupils working in computer environment with Mac Draw . Those pupils use the psibility to change thickness of lines . It permits to isolate elements of figure and in the same time to keep the whole figure . Pupils working in computer environment with CABRI use the instruction which permits to hide a line . The problem for them is that they cannot keep the hole figure

VIII . CONCLUSION

The use of Mac Draw software on Macintosh allows us to encourage the procedures involving different decompositions of the figures . This is principally due to the possibility of changing the thickness of the lines with a simple instruction . CABRI software permits the appearance of a different type of procedures of resolution . There are the role and the status of the figure which therefore take on a different aspect . The demonstration separates from the figure . The later becomes a support for the use of certain caracteristics , this support is variable in different parts of the resolution of the problem . This was already apparent in Mac Draw environment but is more important in CABRI one by the change of the original figure into another , which is similar for use of Thales's property . In Mac Draw environment difficulties relative to the number of times Thales's theorem has to be used and relative to the number of forms of Thales's theorem to use , decrease . In CABRI environment difficulties relative to the direction of the parallels , the use of Thales's property with secant lines , also decrease . The role of the variables change of space , presence of the use of algebraic measure , and algebraic computations is important but it is similar in all the different environments . CABRI adds new possibilities because it dœs not only give facilities for the decomposition of the figures but also it gives a change of the status of the figure in a resolution of a problem .

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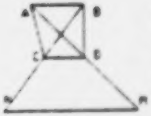
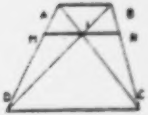
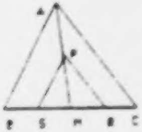


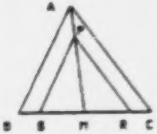
TRAPEZEB
ABCD is a trapeze . M is the image of A in the point symetry with D as center . The intersection of line BC and line parallel to AB including M , is a point called N . Show that N is the image of B in the point symetry with C as center



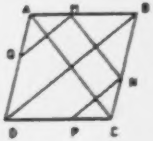
TRAPEZEA
ABCD is a trapeze. The intersection of (AC) and (BC) is a point called I . The parallel to (AB) including the point I is secant to (AD) in a point called M and is secant to (BC) in a point called N
Show that I is the middle of [MN] .



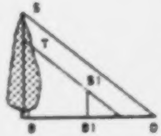
TRIANGLEB
ABC is a triangle . M is a middle of [BC] . P is a middle of [AM] . The parallel to [AB] including the point P is secant to [BC] in a point called S . The parallel to [AC] including the point P is secant to [BC] in a point called R . Show that R is the image of S in the point symetry , the center of witch is M



TRIANGLEC
ABC is a triangle . M is a middle of [BC] . P is a point of [AM] . The parallel to [AB] including the point P is secant to [BC] in a point called S . The parallel to [AC] including the point P is secant to [BC] in R . Show that R is the image of S in the point symetry , the center of witch is M .



PARALLELOGRAMM
ABCD is a parallelogramm . M is a point of [AB] . The line parallel to [BD] including the point M is secant to [AD] in a point called Q . The line parallel to [AC] including the point M is secant to [BC] in a point called N . The line parallel to [BD] including the point N is secant to [DC] in P . What kind of geometrical figure is MNPQ ?



POPLAR TREE
That is a poplar tree BS . BD is the shadow of the poplar tree . That is a stick B1S1 . B1D1 is the shadow of the stick . We have the measure of BD , B1D1 , B1S1 . Calculate the measure of BS (we suppose the sunbeam are parallels)

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PARTITIONING AND GEOMETRY: IS THERE A CONNECTION?

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This paper proposes a model relating the hierarchy of partitioning skills to the van Hiele levels of geometric understanding. A study was conducted to examine the viability of the model by examining a possible correlation between the ability of upper elementary students to partition a figure to obtain a unit fraction and their understanding of geometry as determined by the van Hiele levels.

Purpose of the Study

It appears that the underlying assumptions made by textbook authors in choosing figures are (1) that children focus on the geometry of the shape, and (2) that attaining particular fractional parts is facilitated by shape selection (for example, selecting a triangle for thirds). Pothier and Sawada have found that there is a great deal of invalidity in both of these assumptions.

In a study by Pothier and Sawada (1983), a five-level theoretical account of the development in young children of partitioning as a process serving as the foundation for the child's construction of rational number was proposed. In that study, the tasks used with children were characterized as follows: (1) as concerns shape, the objects were either circular or rectangular; (2) as concerns procedure, the number of equal parts the child was asked to partition the objects into was always specified by the researcher, and (3) as concerns sequence, the order of tasks performed by the children was always the same in that each child dealt with a task that required partitioning into two or four equal shares before having to partition an object into three or five equal shares. A later study (1984) involved the partitioning of "giant cookies", using regular triangular and regular pentagonal shapes. They found the dominant use of parallel or orthogonal cuts to be striking, and that the children used this procedure as a general partitioning strategy without regard for the shape of the object being partitioned. The partitioning mechanisms they found, listed in the order of their dominance, are: halving cuts; orthogonal lines; parallel slicing; truncating vertices.

The van Hiele model is of interest to mathematics researchers in explaining why students have difficulty with complex concepts in geometry, because it partitions the learning of geometry into five distinct levels. These levels are: level 0, Visualization; level 1, Analysis; level 2, Abstraction; level 3, Deduction; level 4 Rigor. It is inherent in the van Hiele theory that, in understanding geometry, a person much go through the levels in order. The levels are different from the Piagetian levels in that the student can move from one level to the next through experience and instruction; the levels are not based upon age or maturation.

Since various geometrical objects occur often in learning materials used for fraction experiences in the early grades, the findings of Pothier and Sawada that "children seem not to attend to the geometric properties of the shapes" (1984, page 1) question whether there is any valid rationale for including and sequencing these types of activities. For this reason, a systematic examination of children's partitioning behavior of tasks involving the many geometric shapes often included in early fraction experiences should be undertaken.

Since the process of partitioning consists of dissecting a shape into equal parts, the notion of congruence is relevant. Such children are in need of developing a stronger awareness of congruence. One question of interest is whether students had an understanding, either intuitively or more formally, of congruence. Children seem to partition as if they were oblivious of congruence and thus impose mechanisms which violate the geometry of the shape. Such children are in need of developing a stronger awareness of congruence. Apparently, the geometry curriculum experienced by children up to grade three is insufficient to develop the needed awareness. Further research into congruence is called for.

The Method

In order to investigate this proposed model, a study was conducted with 34 students (11 in grade 4, 12 in grade 5, 11 in grade 6) in the gifted class of an elementary school in Rochelle, Illinois, to determine if an informal understanding of geometric properties enhances their performance in partitioning tasks. The students' levels of partitioning were determined using the partitioning tasks developed by Pothier and Sawada, and their van Hiele levels of geometric understanding were measured by tasks derived from the interview questions validated by Mayberry. The students then studied a unit on polygons using Logo geometry, in conjunction with a concurrent study. Subjects were then posttested at the end of the unit, both as to their partitioning levels and van Hiele levels. We examined data to establish (a) if there was an initial correlation between their levels of partitioning and van Hiele level and (b) if the instruction in geometry had an impact on their level of partitioning.

The average length of each interview was about 25 minutes. During an interview the child was presented with a series of regular figures on tagboard, and a supply of straws, and challenged to use these materials to partition the figure in various amounts. After a 20-day unit on polygons using Logo geometry, the subjects were interviewed again. When beginning the interview, the child was told that we would be looking at shapes representing cakes or giant cookies, and that we wanted to cut them into pieces so each person in a group could have a piece, and so that everyone would have the same amount. For the rectangle and circle tasks, the students were directed to partition in the following order: 2,4,8,6,3,5. For the remaining tasks (regular triangle, pentagon, pentagram, hexagon), the students were asked for what size group they could share the figure; if the student did not partition with a quantity corresponding to the number of sides of the figure, that quantity was specifically requested by the interviewer.

The Data

Four questions provided the foci of the investigation. Obviously the questions are not only interdependent, but are also very large in their scope.

1. Is there a relationship between the level of partitioning, as defined by Pothier and Sawada, on which a child partitions, and the level of understanding demonstrated in geometry, as measured by the van Hiele levels?

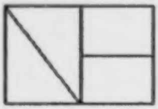
Analysis has indicated that there seems to be a correlation between students performing on partitioning levels 2 and 3, and performing on van Hiele level 0, or below zero. Currently we are investigating the type of change, and on what specific tasks.

2. Will studying geometry impact a student's performance on partitioning tasks?

Nearly all of the students evidenced a change in their partitioning approaches on at least one of the figures, and frequently on many of the figures. The data is currently being studied to pinpoint what types of changes are occurring, and to determine whether there is a pattern to the changes.

3. Does the notion of congruence play a part in students' understanding and/or actions in partitioning?

At the end of the pretest, students were given the rectangle and asked to share it evenly between himself/herself and the interviewer. He/she was then asked to share his/her half with a friend. Both of these cuts were usually vertical or horizontal. The interviewer then divided her half diagonally. The resulting figure looked like:



The student was then asked if these pieces were distributed, would each of the four people receiving a piece have "the same amount." Only two students indicated no; one student indicated that the triangles looked bigger; the other referred to the fact that the pieces were of different shapes. Of the students answering yes, most utilized an explanation involving the phrase "each person has half of a half."

However, at the end of the posttest, each student was asked to demonstrate as many ways as possible to cut the rectangle so that it could be shared among four people. Not one student utilized this same partitioning. The only partitioning involving non-congruent pieces was the one obtained by cutting across the diagonals. Most students did not examine their resulting partition, and it is surmised that they were not aware of the fact that the pieces were not congruent; one student did obtain this partition, and then rejected it because the pieces did not look the "same" to her. Clearly, these students felt that a partition involving "the same amount" required congruence. This may have been impacted by their lack of concept of area.

4. Does the notion of symmetry play a part in the students' understanding and/or actions in partitioning? Specifically, do students use lines of symmetry in order to aid their partitioning task?

No student had difficulty partitioning these figures so that they could be shared among two people. The predominant strategy was to use a vertical line of symmetry; less frequently, but still commonly used, was a horizontal line of symmetry. Other lines of symmetry were virtually ignored, complicating the partitioning task. One student asked could he "move" the figure, and in this way obtained a "new" vertical line of symmetry. Apparently the location of the figures appear "fixed" to the students, and they are not utilizing information they know (or could infer) from the lines of symmetry. It remains a question as to whether having more experience with the lines of symmetry would affect their partitioning.

Summary

Analysis of protocols indicates that most of the students are initially functioning at the lowest levels of both the partitioning hierarchy and the van Hiele hierarchy. Only one student has indicated that she considers the sides and vertices of a figure when partitioning.

Most children in grades 4 through 6 have a good understanding of halves, in the sense that they have the ability to partition continuous embodiments. We were surprised by the extent of the difficulty experienced by fourth graders in finding fourths. As a result of schooling, children had acquired a "half of half" notion, but often they did not use their notion of "halves" to obtain congruent pieces, but retained their algorithm of apply vertical cuts (specifically to the circle). It is only by grade 5 that some children seem to have developed a mental image of the shape of "one-quarter" in certain common continuous embodiments. The lack of understanding of "thirds" was especially apparent. We had thought that thirds was an embodiment which the children would grasp because the triangle had three sides of equal length. However, we were mistaken. Indeed, even at the sixth grade level, most children did not relate "thirds" to the sides of the equilateral triangle, and only about a third of the students interviewed obtained a correct partitioning.

Children don't think about "wholes" or partitioning. Rather, hearing a verbal stimulus (e.g., "one-half"), they consult their long-term memories and evoke the appropriate visual images. [On being asked to share with three people, one child appeared stumped, then said, "Aha! I remember that!" and produced the Y cut.] In most cases, this image has a vertical or horizontal line of symmetry.

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VARIATION IN STUDENT GEOMETRIC CONCEPTS

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Middle School students were asked to draw, define, and choose examples of geometric concepts of triangle, rectangle and square. A written test showed apparent inconsistencies between a student's own definition and the student's choice of examples. Videotaped interviews of 21 students provided insight into reasons behind student responses. Students did not view the responses as inconsistent. Imprecise vocabulary, weak definitions, illogical reasoning and strong prototypes contributed to their perspectives.

Individual middle school students showed a variability within their own responses as they moved through geometric tasks involving drawing, writing, choosing, answering questions and explaining. Students appeared to have a different concept depending on whether they were drawing rectangles, defining rectangles, or choosing examples of rectangles. In several instances the concepts appeared to be inconsistent. Do students develop separate, unrelated concepts about rectangles. Investigation of concept development across contexts offers insight into student responses.

Previous studies involving geometric concepts have reported student behaviors and responses that appeared illogical, inconsistent or unconventional. Elementary school students' thinking was not only inconsistent with conventional concepts but students appeared to be inconsistent within their own thinking (Wilson, 1986). Students' abilities to operate logically may be tied to their cognitive development (Piaget & Inhelder,1964; Winer, 1980) or they may be related to levels of understanding gained through experience (van Hiele & van Hiele-Geldof , 1984; Burger & Shaughnessy,1986). Students form powerful prototypes (Kellogg, 1980; Anderson, 1980) that influence their reasoning. Although explanations for the inconsistencies vary, the existence of inconsistent responses is well documented.

METHODS AND PROCEDURES

Two hundred eighty-eight students from two schools including six 6th grade classes and six 8th grade classes participated in the study. Middle school students were chosen because they had a better mathematical vocabulary than elementary school students and most students were in transition from concrete-operational to formal-operational thinking. These students have had a variety of school and nonschool experiences with rectangles.

A written test was administered by each classroom teacher. The purpose of the test was to look for patterns of behavior in a large number of students. The test also served as a source of reflection by students during follow-up interviews. The test assessed students' concepts about triangles and parallelograms using four components : (1) drawing figures (2) defining figures (3) choosing examples of specified figures and (4) answering questions about figures. Students were permitted to move freely between sections of the test and easily completed the test in about 10 minutes.

Follow-up interviews were given to 21 students representing the 12 different classrooms. The students were chosen because their papers raised interesting questions. The interviews lasted approximately 40 minutes. All interviews were conducted by the researcher and recorded on both audio and video tape. The interviews began with a script but deviated to explore particular responses. Often students were asked how they would teach a particular idea to a younger student. This technique encouraged students to express their own ideas rather that repeating a memorized response.

FINDINGS

The written tests provided information on frequent responses and combinations of responses. The interviews provided an in depth look at the reasoning behind specific responses. While the test data suggested shared misconceptions and frequent patterns of response, the interviews illustrated the diversity between students. There were common inconsistencies and beliefs but each individual provided a different explanation or rationale. In some instances the tests and interviews suggested contradictory interpretations. For example, the test data suggested that students wrote definitions that were completely independent from their choices of examples for a particular concept. However, during the interviews students explained that they did refer to their definitions when choosing examples or vice versa. Interestingly, the students saw consistency in responses that appeared inconsistent from a conventional mathematics perspective. Both the test and the interviews focussed on how students used or worked with their own definitions, drawings, and example choices regardless of how well their answers fit conventional mathematics.

Variation in Student Concepts

Students had difficulty with the relationship of squares and rectangles. Since many people define a rectangle as being long, the idea that a square is not a rectangle is consistent with their unconventional definition of a rectangle. However many students were not consistent. Fifty-four percent of the students (156) wrote definitions that permitted squares as examples of rectangles, but only 17% chose a square when asked to circle all the rectangles in a set of polygons including 2 squares. Only 8% of the students drew a square as an example of a rectangle. Ninety-two students (32%) agreed to the statement that all squares are rectangles and yet half of those students did not circle squares as examples of rectangles. More than half of the students were not consistent between definitions, examples, drawings and answering questions with respect to rectangles.

Students were attentive to angles in some contexts but not in others. Reference to angles was missing from more that 75% of the definitions, but students seem to pay attention to angles when they chose examples or drew pictures. Two hundred twenty students agreed to the statement "All figures that have 4 sides with the same length are squares." and yet half excluded rhombi without right angles as examples of squares. In the eighth grade, 106 students did not require rectangles to have right angles in their definitions and yet only 61 chose parallelograms without right angles as examples of rectangles. Sixth graders operated in a much more consistent manner with 122 students omitting right angles from their definitions of rectangles and 106 students choosing nonrectangular parallelograms as rectangles.

Reasons for Variation

Interviews with students provided insight into some of the reasons for variety in student responses related to the same concept. Student reasoning was influenced by the following factors.

Student definitions were often incomplete reflecting students lack of concern for necessary and sufficient conditions. One student claimed that it was not necessary to include right angles in the definition of rectangle because everyone knows that rectangles have right angles. Another student expressed a concern for making definitions short.

I: Do you think that sometimes we think things more than in the definition?
S: Oh, I do!

I: Can you give me an example?
S: Sometimes we are writing down definitions, just out of the spelling book and I won't put the whole thing.

I: Oh, kind of a short cut?
S: yeah.

I: Why don't you write the whole thing?
S: Cause, I don't want to write anymore. And just try to make it shorter if you can. And don't make it so long.

Student definitions used imprecise terms allowing the same term to vary in its meaning. The ambiguity of words was useful to students since they could explain inconsistent ideas by changing the meaning of the words. An eighth grader defined a rectangle as a closed figure where not all sides are the same length." During the discussion it was clear that the student wanted the opposite sides the same length and adjacent sides different lengths. The author was satisfied that his definition meant what he intended. Words and ideas had a temporal component allowing flexibility. For example, an eighth grade student defined a rectangle as a "4 sided figure with all sides the same - sometimes." Consider the flexibility of the following definitions for a rectangle.

Eighth grade: A longer shape with 4 long straight lines
long, 2 even and 2 even
a geometrical figure which lenth and width are not equal
4 sided, sides are parrallel

Sixth grade: It looks like a long stretched out box with 4 sides
a rectangles is long, and has 4 sides equal
two long lines across ways connected with two shorter ones going up and down with 4 points

Students worked from a prototype producing a visual image of the concept that was more salient in choosing examples and drawing examples than in writing definitions and answering questions. In the following discussion, a student defended his choice of a nonsquare rectangle as a rectangle.

I: Tell me what was special about it that made it a rectangle.
S: Well, these two sides were a lot more longer than these 2 sides were.

I: O.K. Now was that part of your definition that 2 sides be longer than the other 2 sides?
S: No, I didn't do that. That's one thing I missed.

Justifying a discrepancy between his definition and choices of rectangles, one student explained "those are the words you write" but this is what a rectangle really looks like. Students' prototypes were flexible enough to allow changes in orientation and size. Students were also able to choose and draw rectangles in different orientations, but the most common figure was a long rectangle oriented with the longest side parallel to the bottom of the page.

Students did not make conventional mathematical assumptions. For example one of the interviewed students did not assume that the figures were static maintaining constant properties. He pictured dynamic figures that stretched and changed properties. When the interviewer expressed confusion over the student's claim that rectangles are squares and squares are rectangles, the student explained

S: Well, the rectangles can be pulled out. The sides can be pulled out to make it look like a square. And the square can be pulled out to make it look like a rectangle.

Some students were willing to hold two contradictory ideas. It was not always clear if the ideas were contradictory to the student, but in the following conversation it appears that the student is aware of the contradiction in the response but still maintains her position. The interviewer and the student are discussing the statement "All squares are rectangles." The student agreed that all squares are rectangles but she struggled with the conflicting idea that rectangles are always long.

I: So some squares could be rectangles?
S: Yes, all squares could be rectangles.

I: . . . now if you went back to this page where you chose rectangles, would you choose any different ones this time? (She did not choose squares as examples of rectangles.)
S: I don't think so. I think all rectangles are long and you usually think of a square as small.

I: . . . So you won't choose these (squares) as rectangles but you are willing to put back here (in the test) that squares are rectangles. (She nods yes.) So you are going to agree there but you aren't going to choose them here, so I'm confused.
S: I just circled the longer ones because most people think of the longer ones.

I: If we did it over again would you still just circle the long ones?
S: Yes.

Students did not consider logic or deductive reasoning as useful tools to defend their decisions. Even counterexamples were not used. In a discussion about the relationship of squares and rectangles, one student tried to "remember" claiming I know it is either "Squares are rectangles." or "Rectangles are squares." When ask if we could figure it out, he was definite that you couldn't figure it out ; "You just have to remember." Even students who agreed with the statement "All squares are rectangles." seemed to remember and only a few could explain why it was true. Some students used deductive reasoning but made faulty deductions. One student explained that a rectangle could be divided into two congruent triangles by drawing the diagonal. She argued that parallelograms were rectangles because if you drew the diagonal of the figure, you would create two congruent triangles.

DISCUSSION

The findings support Piaget's position that middle school students have not attained formal reasoning. The data also confirm van Hieles' assertion that students who are operating at the holistic level do not understand enough about properties to build relationships or use deductive reasoning. However, the findings question whether researchers can determine a level of development or understanding from a student's responses in only one context. Students exhibited different knowledge about properties of a rectangle (e.g. angle, parallel, equal sides) when they were choosing examples or drawing than when they were writing definitions or answering written questions.

Mathematics teachers and researchers rely on definitions and examples as they develop new ideas or try to communicate mathematical ideas. Unfortunately, many mathematics learners are not able to utilize the power of a definition or understand the relationship of definitions and examples. As researchers and teachers, we frequently assess a students understanding of a concept using one context. It is common to ask a student to "provide an example", "give a definition", or "state in your own words", but any one of these tasks is not sufficient to uncover the variability in the student's concept as he or she operates with the concept in different contexts. There is some evidence that the student may actually be operating with one concept but from an outside perspective we can not understand the complex relationship of how the student perceives the concept in different contexts. In teaching we need to use varied contexts in order to stimulate learning and to offer students the opportunity to struggle with a concept from different perspectives. As researchers, we need to assess student concept development from a variety of contexts in order to understand the complex process.

RESOURCES

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EFFECTIVENESS OF SMALL GROUP PROBLEM SOLVING WITHIN CLASSROOM SETTINGS

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Six groups of seventh-grade children participated in a problem-solving activity involving the construction of a lattice structure. Analysis of student work, videotapes of the sessions, and observer notes revealed students were successful in (1) constructing solutions, (2) recognizing the meaning of the various structures, and (3) understanding of the relatedness between this representation, other problems encountered, and numerical situations.

Procedural understanding of mathematical concepts depends on active engagement in problem situations where the learner is able to connect the symbolic forms to representations that correctly embody their structure. This ability develops as the child constructs a variety of representations with which to consider the structure of particular mathematical concepts (Dienes, 1960). Models are built from concrete objects, represented visually as drawings, and interpreted with symbols, both written and spoken, that remain invariant across problems (Bruner, 1973). These various representations, including real-world situations, are not linear in their development, but offer the learner a repertoire of models on which to base strategies for analyzing and solving classes of problems (Lesh, Landau, & Hamilton, 1980).

According to Cobb (1988), development of understanding is enhanced by interaction with others, both teachers and peers. The process of constructing solutions to particular problems is negotiated through attempts to verbalize and defend strategies and representations.

Earlier work examining problem-solving activities done by small groups of children indicates that learning does occur for individuals. Analysis of the interactions within these groups provides a rich source of data for consideration of the process of children's problem solving, the strategies that they employ, and their construction of understanding of particular mathematical structures (Alston and Maher, 1984: Maher, Alston, & O'Brien. 1986).

This investigation describes the problem-solving activity of small groups of children working together on a problem task which used a concrete nonnumerical embodiment to represent and explore the presence or absence of certain properties of a binary operation on a set of elements: namely, closure, commutativity, identity, and inverse elements. Particular problem solving behaviors that were studied include: (1) strategies employed in constructing solutions, (2) generalization from the particular problem activity to other representations, particularly numerical ones, and (3) development of individual solutions and the group's consensus as a result of interaction among the members.

Methods and Procedures

The study included eighteen 12 to 13 year old children, nine boys and nine girls, who were members of seventh-grade mathematics classes in schools in two communities - one an independent K-12, and the other a K-8 public school.

The eighteen children, along with their classmates in five different mathematics classes, were organized by their teachers into groups of two or three on the basis of similar ability and potential compatibility for working together. Although the children were permitted as much time as they needed to complete the problem task, each of the groups finished within two 45 minute class sessions. Two video cameras were used within each of the classes to follow the problem-solving activity of particular groups of children. The six groups considered in this study, representing a range of ability from average to above average according to class achievement, were video taped. A trained graduate student served as observer and recorder for each group during the problem activity.

One of the groups was made up of three boys; two groups included one boy and two girls; two groups included two boys and one girl; and one group was made up of three girls.

Instructions and record sheets for the problem task were given to each of the children along with the cards that were to be used to model its solutions. The children were instructed by the teacher to work together to figure out solutions and were asked to choose a recorder to report the concensus solution, when possible. Also, the children were asked to record their own solutions. The teacher's role during the sessions was to respond to questions of clarification. They were instructed not to intervene in the children's construction of solutions.

The Problem Task

The task used in the activity was adapted from an exploration involving properties of sets from the School Mathematics Project, Book D and is described as follows:

> **Five cards, denoted as A,B,C,D, and E, each with a different polygonal shape cut out from the center, constituted the elements of a set with the operation defined as putting one card on top of another and the result being the hole formed by the two cards. Part 1 of the task directed the children to use cards A, B, C, and D to complete a 4 x 4 table for the set with the defined operation. These four cards formed a closed, commutative lattice structure. The children were asked to consider a series of**

questions about closure, the existence of identity and inverse elements, and commutativity within this set. Card D contained a rectangle which enclosed the shapes cut from the other cards, served as an identity element, and was its own inverse. No other cards had inverses. Part II introduced a fifth card, E, and asked for a 5 x 5 table. This set was not closed under the operation but the other characteristics remained the same. Part III instructed the children to create a different set of five shapes that would be closed under the operation "place on top of". Part IV asked the children to reflect on the problem task, to indicate what they thought was important in constructing their solutions, and to note any other problems or ideas that this task brought to mind.

The videotapes of the sessions were viewed independently by two or more graduate students who were instructed to identify and excerpt segments from the tape that indicated for individual children the following:

a. success in solving each part of the problem;
b. use of the concrete representation to answer parts of the question;
c. solutions to parts of problem constructed through reference to patterns, the tables, or without using the cards;
d. generalization to other problems;
e. generalization involving numbers;
f. active engagement in the task;
g. interaction with other children in constructing solutions;
h. assumption of role as leader;
i. construction of solutions by working alone;
j. submission of ideas to more dominant group member;

The analyses of the tapes, along with observers' notes from the sessions, and children's worksheets provided data for this report.

Results

The Table provides a summary of the results according to the criteria indicated for the videotape analyses. Limited space precludes full documentation of strategies employed, solutions constructed, or generalizations made by the children. However, examples are provided as illustration.

Table: Documented Problem-Solving Behavior

		Criteria									
Group	Child	a	b	c	d	e	f	g	h	i	j
I	1	x	x	x	0	0	x	x	x	0	0
	2	x	x	x	0	0	x	x	0	0	x
	3	x	x	x	0	0	x	x	0	0	x
II	1	x	x	x	x	x	x	x	x	0	0
	2	x	x	x	x	0	x	x	0	0	x
	3	x	x	x	0	x	x	x	x	0	0
III	1	x	x	x	0	x	x	x	x	0	0
	2	x	x	x	0	x	x	x	x	0	0
	3	x	x	x	0	0	x	x	x	x	0
IV	1	0	x	x	x	0	x	x	x	0	0
	2	0	x	x	0	0	x	x	x	0	0
	3	0	x	x	0	0	x	x	0	0	x
V	1	x	x	0	x	x	x	x	x	0	0
	2	x	x	0	x	x	x	x	0	0	x
	3	x	x	0	x	x	x	x	0	x	0
VI	1	x	x	x	x	x	x	x	x	x	0
	2	x	x	x	x	x	x	x	0	0	x
	3	x	x	x	x	x	x	x	x	0	0

Note: x = presence of criteria, 0 = absence

1. Strategies and Solutions:

Only one child filled out the charts with no physical reference to the cards. After completing his chart working alone, he compared his conclusions with those of the group members. Five of the six groups considered the results of the first chart to be a subset of the second and built their solutions accordingly. In Group I, the leader noted that each card combined with itself remained unchanged and led his group to fill in the diagonal of the chart first. He then convinced the other two boys that "C on

anything was C and that D on anything was the other thing".

In deciding that Card D was the identity, one child in Group III referred to zero **"because zero added to any number resulted in the other number".** Two other groups chose Card D after considering how both Card C and Card D operated within the set. A member of Group II stated: **"It's C because C is smallest and fits evenly into the others."** A second added: **"Then D works also, because it fits over the other shapes and doesn't change them."** After rereading the questions, they agreed on the statement: **"No - it's D because C changes the shape of the others."**

Groups I and IV first responded that no elements had inverses. Group III first asserted that any card when paired with itself would give D. Groups I and III revised their conclusions while Group IV did not.

2. **Explanations and Reflections:**

Two groups responded that this problem reminded them of no other that they recalled, stating that it was **"just working with shapes".** Two groups responded that the problem was like addition of numbers. One boy said: **"It's like 3 + 1 = 1 + 3. Like adding".** Two other groups responded that the problem reminded them **"of fractional parts fitting into a whole".** One of these groups also was reminded of **"multiplication patterns"** and the final group stated that the problem reminded them of **"problems about numbers, like dividing numbers into one large number."**

Conclusions

Strategies varied among the groups in constructing solutions as did the children's explanations and reflections, from specific references and use of the physical objects to generalizations about pattern and numerical operations.

All of the children seemed to be actively engaged in the task, participating in the group's construction of solutions as well as deciding about their own.

Insights gained from the children's problem solving and the feasibility of such activities for classroom instruction suggest the value of further research concerning their appropriateness as activities for developing children's structural understanding of these concepts within natural classroom settings for school mathematics.

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Cognitive Controls: Are They Related to Mathematics Problem Solving Performance?

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This study investigated the relationship between mathematics problem solving performance and cognitive controls. 158 seventh and eighth grade students completed the Iowa Problem Solving Project (IPSP) test, seven cognitive control measures from Letteri's Cognitive Profile battery, and four measures of simultaneous and successive coding. Regression analysis indicated that the best predictors of problem solving performance were the analytical and reflective cognitive control measures and the simultaneous coding measures.

Introduction

Researchers recognize the importance of cognitive controls during the mathematics problem solving (MPS) process. Burton (1984) and Schoenfeld (1985) have suggested that teaching heuristics to improve MPS performance is not adequate. They suggest that attention needs to be given to the cognitive controls that make decisions about the processes taking place during problem solving. But to utilize cognitive controls to enhance the teaching of MPS, we must develop an objective description of the cognitive controls involved.

Measurable Cognitive Processing Model

Letteri (1979) has developed a model of seven cognitive style dimensions that he refers to as cognitive controls related to academic performance. The seven dimensions are:

1. Tolerance (vs intolerance) is the skill required to monitor the modification of cognitive structures.
2. Analytical (vs global) is the skill required to segment complex information into component parts for identification.
3. Focusing (vs nonfocusing) is the skill required to attend to one relevant component without being distracted by others.
4. Reflective (vs impulsive) is the skill to perform a highly accurate comparison between two or more pieces of information.

5. Broad (vs narrow) is the accuracy/consistency used to judge category placement of new information in long term memory.
6. Complex (vs simple) is the arrangement and integration of information into cognitive structures in long term memory.
7. Sharpening (vs leveling) is the skill to maintain distinctions between structures and to avoid confusing or overlapping them.

Letteri (1979) reports that more than 60% of the variance in standardized achievement test scores can be explained by the seven cognitive controls in his model. He argues that the multi-dimensional approach provides information relating the combined impact of a configuration of cognitive dimensions on various criterion variables. Letteri (1982) has also reported a quasi-experimental study using his cognitive control model for low achieving mathematics students; six months after cognitive augmentation, the experimental group showed an increase in performance that was four times the control groups increase. Thus, Letteri's cognitive controls appear to be related to mathematics performance.

But theoretical models for competence in MPS suggest that Letteri's model is not adequate. Goldin (1987) has hypothesized that a planning language controlling both a verbal/syntactic process and an imagistic process are necessary for success. He argues that the three components work together to elicit the formal notational solution to a problem. Letteri's model only delineates the internal cognitive controls of the planning language. We need a way to look at both the verbal/syntactic process and the imagistic process.

Luria's (1973) neuropsychological model appears to offer the missing components. In Luria's model, a planning component controls two coding components: simultaneous processing (similar to imagistic processing) and successive processing (similar to verbal/ syntactic processing). Researchers have consistently shown a relationship between mathematics performance and the coding processes (Das, Kirby,, & Jarmen, 1979). Unfortunately, the same has not been true for the planning component; different methods of measurement have shown inconsistent relationships between mathematics performance and planning (Das, 1984; Garafolo, 1986; Hunt & Randhawa, 1983). Das (1984) has suggested that the planning component is related to cognitive style and may not be unidimensional. This suggests that a multi-dimensional model such as Letteri's could be used to measure the planning component.

Letteri's model with the addition of simultaneous and successive processing from Luria's model provides a measurable model similar to Goldin's competence model.

Methodology

The purpose of this study was to determine the relationship between mathematical problem solving performance and Letteri's cognitive controls plus simultaneous and successive processing. The subjects were all seventh and eighth grade students from a private middle school in the New England area. The students were assessed during four class sessions and one individual forty-five minute session using twelve different tests. The Iowa Problem Solving Project Test, a 30-item test measuring different steps of the problem solving process, was used to measure MPS performance. The seven cognitive style measures of the Letteri Cognitive Profile battery were used to measure an individual's preferred mode of processing information on each of the seven control dimensions. Simultaneous coding was measured by a geometric matrices test and the Memory-for-Designs test; and successive coding was measured by a word recall test and a digit span test. (For details see Biggerstaff, in press.) Scores were composited to form a single score for problem solving performance, for each cognitive control, and for each coding process. The data were analyzed using multiple regression techniques with an emphasis on finding the best parsimonious model of cognitive controls for predicting scores on the IPSP test.

Results

A base model was established by regressing grade, sex, and/or mathematics class variables on the IPSP scores. Neither grade nor sex differences were statistically significant. However, a variable identifying the thirteen students in the lowest seventh grade mathematics class (LMC) did have a significant effect in predicting IPSP scores (Model I in Table 1.). The average score for the LMC students was 7.7 points below the average score for all other students. Thus, Model I was used as the base model.

Table 1.
Significant Regression Models

Mod	LMC	Toler	Analyt	Focus	Reflect	Narrow	Complx	Sharp	Simul	R2
I	-7.9****									.345
II	-7.1****	.18	.22**	.01	-.28**	-.004	.03	.83		.456
III	-7.2****		.22***		-.30***					.448
IV	-6.9****		.14*		-.29***				.15*	.478

* p < .05, ** p < .01, *** p < .001, **** p < .0001

To test the hypothesis that the seven controls could explain a large proportion of the variance in MPS performance, all seven control measures were added to the base model (Model II in Table 1.). The seven control measures and LMC explained 45.6% of the variance in the IPSP scores. Only two control measures, analytical and reflective, had coefficients significantly different from 0. A regression equation including only the analytical and reflective control measures explained 44.8% of the variance in the IPSP scores (Model III in Table 1.) . Thus, the inclusion of the other five control measures was not significant in the regression equation. The best parsimonious regression equation for the cognitive controls was:

$$IPSP = .22\ Analytical - .30\ Reflective - 7.18\ LMC + 21.89 \qquad [1]$$

To test the hypothesis that the two coding measures would improve the explanation of the variance in the IPSP scores, each coding measure was added to Equation 1. The successive coding measure was not statistically significant in predicting the IPSP score, while the simultaneous coding measure was significant. The addition of the simultaneous measure to Equation 1 resulted in 47.8% of the variance in the IPSP scores being explained (Model IV in Table 1). Thus, the best regression model for predicting IPSP scores for this sample was:

$$IPSP = .14\ Analytical - .29\ Reflective + .15\ Simultaneous - 6.89\ LMC + 19.98 \qquad [2]$$

It should be noted that the simultaneous coding measure was significantly related to three of the cognitive controls: analytical ($r=.42$, $p=.0001$), sharpening ($r=.21$, $p=.007$) and complex ($r=.16$, $p=.05$).

Discussion

The results suggest that the development of analytical skills, reflective skills, and the simultaneous coding process for this sample of students may enhance MPS performance. The results did not indicate that all seven control measures were necessary for success in MPS as measured by the IPSP test. Also, the variance in the IPSP score explained by the cognitive controls was less than expected from Letteri's reported results. This may be the result of two factors. First, the IPSP test is not a typical standardized achievement test. It requires students to use higher order thinking skills rather than applying practiced mathematical algorithms. Second, the sample is not a random sample of seventh and eighth grade students. The expected difference in the average IPSP scores of the seventh and eighth grade students did not occur. The results indicate the cognitive controls and coding processes that might predict success in MPS for a higher ability group of students.

Since the analytical and reflective cognitive controls were the best predictors of the IPSP score, we could conclude that empha- sizing the development of those two cognitive control dimensions would have the greatest impact on improving MPS performance. The analytical control skill would help students break a complex problem into more manageable pieces as well as help them establish subgoals that had to be completed before the problem could be solved. The reflective control skill would help students develop a more systematic approach for examining and solving a problem. In general, students would be encouraged to complete a more exhaustive search of viable solutions before they give up.

The development of the simultaneous coding process would also appear to be important for success in MPS. The development of simultaneous processing would improve students abilities to identify patterns and create mental pictures of the verbal information in order to understand complex relationships. Students would be encouraged to begin the problem solving process by conceptualizing the subgoals necessary to solve the problem in relationship to a general goal state.

Though these skills appear to be related to heuristics that have been taught in mathematics classrooms, they are different. The cognitive skills should be taught as a general planning language that the students can use for higher order thinking tasks and readily transfer to the mathematics problem solving process. Then students will have control of the thinking processes necessary for success in math problem solving.

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TEACHING EXPERIMENT EXAMINING THE RELATIONSHIP BETWEEN OPERATIONAL LEVEL AND GENERALIZING ABILITY IN THE CONTEXT OF MATHEMATICAL PROBLEM SOLVING

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A Krutetskii-like teaching experiment was conducted to examine the relationship between Piagetian developmental level and the ability to generalize in mathematics. It was concluded that developmental level should not be used to preclude instruction in mathematics requiring ability to generalize.

The concern of educators that students be able to generalize their learning is at the core of pedagogical and curriculum concerns. The need for students to form a general concept from specific cases and to recognize the generalization that is embodied in a particular case is essential for meaningful learning; certainly this is the case in mathematics. Educators and psychologists have spent decades conducting research into the generalizing abilities of humans. This research has explored methods of improving generalizing ability and attempted to identify characteristics of good and poor generalizers.

More recently, the developmental levels associated with Piaget's theory have been prominent in research into the readiness of students to learn specific mathematical topics. There is some indication (Schleser, Meyers, & Cohen, 1981) that subjects performing at the formal operational level are better able to generalize in non-mathematical contexts. The purpose of this research is to determine whether this is also true in a mathematical context.

METHOD AND PROCEDURES

Forty-one seventh grade students from a suburban central New Jersey public school system were selected for the study on the basis of their performance on two administrations of a combinatorial logic task. Fourteen students were classified as formal operational, 16 as concrete operational and 11 (who did not meet the criteria for either category) as "unclassified." The combinatorial logic selected to classify subjects into Piagetian developmental levels was an electronic analog of the chemical liquids problem (Inhelder & Piaget, 1958). Earlier studies (DeLuca, 1977, 1979; Sills & Herron, 1976) have shown that this task could be used as an equivalent of the chemical task as a measure of formal thought.

The study consisted of two phases; a Teaching Component and a Problem Solving Component. The Teaching Component provided subjects with activities directed towards the acquisition of the ability to write mathematical expressions using variables (they had received no prior formal instruction in this skill). The Problem Solving Component consisted of several mathematics problems designed to measure the ability to generalize; the solution was a mathematical expression containing variables. The subjects individually met with the clinician for two 30 minute interviews. The first session included the Teaching Component (10-20 minutes) and as much of the Problem Solving Component as time permitted; the second interview was dedicated to the Problem Solving Component. This report is based on the results of Problem 1.

The Teaching Component. The purposes of the teaching component were: (1) to provide subjects with individual instruction in reading and writing mathematical expressions containing variables, and (2) to evaluate their performance on problems calling for this skill. A teaching booklet consisting of 19 problems contained 10 that called for solutions to open sentences with one or more variables and 9 that required students to write a literal or numerical expression (e.g., $x + 3$) to word statements. Subjects who were unsuccessful on an item were then asked a series of questions to assist them in building meaning. The clinician did not provide the correct answer, but rather used questions and inductive methods of instruction to assist the student until a correct response was offered.

The Problem-Solving Component. This component had as its objective to ascertain students' generalizing ability in the context of mathematical problem solving. The problem used for this study was selected from Krutetskii's (1976) study. It consisted of five variants that were identical in form but differed in levels of generalization (see Table 1).

Table 1: The Problem

1e. A notebook costs m cents and a pencil costs t cents. How much would be paid for d notebooks and e pencils?
1a. A notebook costs 50 cents and a pencil costs 10 cents. How much would be paid for 3 notebooks and 4 pencils?
1b. A notebook costs 50 cents and a pencil costs 10 cents. How much would be paid for 3 notebooks and e pencils?
1c. A notebook costs 50 cents and a pencil costs 10 cents. How much would be paid for d notebooks and e pencils?
1d. A notebook costs 50 cents and a pencil costs t cents. How much would be paid for d notebooks and e pencils?

Each subject was given scratch paper and a card with variant 1e and asked to read the problem aloud and solve it. After giving an answer, the subject was asked to explain how the solution was obtained. If the answer was correct, the subject was finished with the problem. In the event the solution to 1e was incorrect, the subject was given variant 1a and asked to read and solve it. If the subject was unsuccessful on variant

1a, the problem was terminated; otherwise variant 1e was given a second time. If successful this time on 1e, the subject was finished with the problem. If 1e was not solved correctly, the student was then given variant 1b. This process continued until the subject was able to solve the *e* variant, unable to solve variant *a*, *b*, *c* or *d*, or able to solve variants *a*, *b*, *c*, and *d* but unable to solve *e*. Thus, the order of the problem was; *e*, *a*, *e*, *b*, *e*, *c*, *e*, *d*, *e*. The measure of the generalizing ability of the subject was determined by the number of intervening variants solved before the *e* variant was correctly answered (if it was).

RESULTS

The successful subjects (those correctly solving variant 1e) were placed into three groups: (a) subjects spontaneously successful (*n* = 6), (b) subjects successful after at most one intervening variant was correctly solved (*n* = 28), and (c) subjects eventually successful (*n* = 33). (Note that all subjects in the first group were in the second group, and that all subjects in the second group were also in the third group.) Characteristics of the groups that were of interest for providing some explanation of success were developmental level and performance on relevant portions of the Teaching Component. Inasmuch as the correct solution to the problem, *md* + *te*, required the subjects to provide expressions involving the operations of addition and multiplication, performance on Questions 11, 12, 13 and 14 of the Teaching Component were included in the analysis. The questions are given in Table 2. Subjects were considered successful on Questions 11, 13 and 14 if their initial responses were correct; otherwise they were considered unsuccessful. It was determined that performance on Question 12 would be characterized as successful for those subjects that had no more than one incorrect response and no more than one clinician initiated question set (induction cycle) prior to the provision of a correct response. Subjects that had two or more incorrect responses but fewer than two induction cycles were characterized as "delayed successful"; those with two or more incorrect responses and two or more induction cycles were characterized as unsuccessful. Two-way Pearson chi-square tests of independence, with the Yate's correction factor used where appropriate, were conducted between each independent variable (i.e., developmental level and Questions 11 through 14) and each of the three categories for success on the problem. Additionally, a step-wise non-linear logistic regression was used to model the probability of success on the problem as a function of one or more of the independent variables.

Table 2: Questions From the Teaching Component Included in the Analysis

11. Sue has 2 doughnuts. Jim has 3 more doughnuts than Sue. How many doughnuts does Jim have? (2 + 3 =) 5
12. Sue has *x* doughnuts. Jim has 3 more doughnuts than Sue. How many doughnuts does Jim have? *x* + 3
13. Bill has 7 dollars. Mary has 3 times as much money as Bill. How many dollars does Mary have? (3 x 7 =) 21
14. Bill has *b* dollars. Mary has 3 times as much money as Bill. How many dollars does Mary have? (3 x *b*)

The two-way analyses supported a relationship between success on the problem after at most one variant and performance on Question 14 (*p* = .0028), and between eventual success and the factors of developmental level (*p* = .0581) and performance on Questions 12 (*p* = .0285) and 13 (*p* = .0024). The logistic regression prediction models contained the following independent variables: developmental level (*p* = .070) for spontaneous success;

Questions 13 (*p* = .078) and 14 (*p* = .000) for success after at most one variant; and developmental level and Questions 12 through 14 for eventual success (the improvement chi-square *p* values were .15, .133, .031, and .001, respectively.

The final answer given by each of the unsuccessful subjects and the number of correct and incorrect applications of arithmetic operations are given in Table 3. Each variant of the problem had a total of three applications of addition or multiplication (e.g., *m* x *d*, *t* x *e*, and *md* + *te*). From the total of the 24 applications for the eight unsuccessful subjects, 16 involved variable expressions and 8 were purely numerical. For seven of the eight numerical instances there is a one-to-one correspondence between performance on the particular operation in the problem and performance on the question in the Teaching Component using the same operation (i.e., Question 11 or 13). Of the 16 cases in which variables were used,

Table 3: Final Answers of Subjects Unable to Solve the Problem

Subject	Final answer	Correct answer	Arithmetic instances			
			Numerical		Variable	
			C	I	C	I
3	$1.50,*e*	$1.50+10*e*	1	0	0	2
7	$1.50,*e*	$1.50+10*e*	1	0	0	2
12	$1.50+*e*	$1.50+10*e*	1	0	1	1
25	$1.50+*e*	$1.50+10*e*	1	0	1	1
8	$1.50,10*e*	$1.50+10*e*	1	0	1	1
15	3+*e*	$1.50+10*e*	0	1	1	1
31	$1.50x*e*	$1.50+10*e*	1	0	0	2
39	d+e	$1.50+10e	0	1	1	1

Note. C = number of correct instances; I = number of incorrect instances.

there were 11 errors (see Table 3). For 8 of these 11, the associated question (i.e., Question 12 or 14) was also initially answered incorrectly.

CONCLUSIONS AND IMPLICATIONS

Developmental level was included in the logistic regression model for spontaneous success. Although this reflects the fact that twice as many formal operational subjects (28.6%) compared with concrete operational subjects (12.5%) who were spontaneously successful, the small number of successful subjects in each category (4 vs. 2) precluded a valid inference. Developmental level was also included in the logistic model for eventual success (all 14 formal subjects were eventually successful as compared to 12 of the 16 concrete subjects), but the improvement chi-square p value of .15 was marginal. Additionally, the minimum expected cell frequency of the two-way Pearson chi-square for developmental level versus eventual success was too small (2.15) to make the results useful. The fact that 75% of the subjects categorized as concrete operational and 64% (7 of 11) of those categorized as unclassified were eventually successful led to the conclusion that developmental level (for concrete operational behavior and above) should not be used to preclude instruction which requires the formulation of mathematical generalizations.

Questions from the Teaching Component were found to be significantly related to performance on the problem for success after at most one variant and eventual success, and also significantly improved the predictability of the logistic models for these categories of success. Additionally, 97% (32 of 33) of the subjects eventually successful on the problem were also successful on Questions 13 and 14 of the Teaching Component while only 38% (3 of 8) of those unable to solve the problem were successful on both questions. It was concluded that performance on the Teaching Component was related to the ability to generalize (as indicated by performance on the problem). Furthermore, the relatively poor performance of subjects in the Teaching Component was attributed to their difficulty in representing English sentences containing mathematical relationships as mathematical expressions. An implication of this study for teachers of mathematics is that students need first to construct connections between the two representations before being challenged to solve problems that are represented in symbolic algebraic form.

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Cooperative Mathematical Problem Solving and Metacognition: A Case Study of Three Pairs of College-Aged Women

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Three pairs of preservice elementary teachers were videotaped nine times over a period of nine months as they worked cooperatively to solve mathematics word problems. The transcribed protocols were analyzed with particular attention given to identifying and categorizing the metacognitive aspects of the moves made and roles played by the individuals in each pair. The study yielded several observations about the nature of metacognition and cooperative problem solving.

INTRODUCTION

In recent years, problem solving and the promotion of higher-order thinking skills have been identified as major goals for the teaching of mathematics. At the same time, there seems to be a movement throughout the educational community away from traditional teacher-directed or individually-oriented teaching methods and toward the use of more cooperative group work in classroom settings. The growing use of cooperative settings in teaching mathematical problem solving suggests the importance of investigating how the dynamics of working together may affect problem solving.

The Russian psychologist Vygotsky developed a useful theory about how collaboration in problem solving might lead to improved problem-solving ability (Vygotsky, 1978). More recent researchers have extended Vygotsky's work to suggest that it may, in fact, be metacognition which provides the link between social and individual processes. Those researchers who have studied the nature and function of parent-child problem-solving interactions have documented the importance of the monitoring role played by the adult. Children learn by being guided by more competent individuals to assume more and more control for their own work. Vygotsky's work suggests that the cognitive monitoring of individuals may be an internalized, truncated form of social interactions.

To date, most studies of control in cooperative problem solving have focused on small children and adults working together. Although it seems clear that adults collaborating with peers can serve as active external monitors and regulators for each other, very little research has documented the types of monitoring moves made by adults as they work cooperatively together or the types of monitoring roles that they play for each other.

DESCRIPTION OF THE STUDY

The primary aim of this research study was to describe in detail a series of cooperative mathematical problem-solving sessions involving three pairs of college-age women, focusing in depth on the monitoring moves made by the individuals within the pairs. Each of the pairs of women was videotaped nine times over a period of nine months as they solved arithmetic word problems together. Since monitoring actions are inextricably intertwined with an individual's abilities, attitudes, and beliefs, the analyses of monitoring moves were interpreted in light of extensive background data collected on each of the individuals.

Analyses of the paired problem-solving protocols utilized a modified form of the Schoenfeld (1985) episodic parsing scheme, which led to identification of many of the individual moves that served monitoring functions within the protocols. These moves, in turn, were classified into a taxonomy of three broad categories of monitoring moves: **procedure** assessments, state assessments, and self or partner reflections.

Assessments were defined as verbalizations that focused on metacognitive regulation relating directly to the problem solution. In a procedure assessment either the feasibility of a procedure was considered, or a new procedure was suggested. State assessments were statements or questions in which an individual monitored the state of a solution process (asking, for example, "what do we know here?" or "where are we?" or "is this getting us anywhere?").

Reflections were defined as verbalizations that focused on a metacognitive experience relating to an individual's thinking, feelings, abilities, or beliefs. Self reflections were statements (or questions) in which an individual reflected upon her own thinking, feelings, etc., whereas partner reflections were questions or statements about the thinking, feelings, etc. of a partner.

It was considered important to classify monitoring moves not only by type (assessment or reflection), but also by function within the overall course of the problem solution (e.g., organizing function, verifying function, etc.) However, initial eforts to classify individual monitoring moves in this way proved unsatisfactory since when monitoring statements are isolated from their context, categorization by function becomes very difficult. Thus, it was decided to determine the general function of monitoring moves by identifying and classifying entire **monitoring events**: blocks of statements where related monitoring moves were evident. The cognitive-metacognitive framework developed by Lester and Garofalo (Garofalo and Lester, 1985;

Lester, 1985) provided a useful scheme for classifying the monitoring events according to their function within the protocol. The framework details four major types of activities in which problem solvers engage: orientation, organization, execution, and verification.

OBSERVATIONS

The analyses of monitoring moves and events, together with information about the problem-solving background of each of the subjects, yielded several general observations about the nature of metacognition and cooperative problem solving. Three are described here.

On the Importance of Ignoring as a Monitoring Move.

Metacognition, by its very definition, involves awareness--an important aspect of which is knowing when to listen. Much less frequently discussed or recognized, however, is the fact that knowing when to ignore is probably just as important a metacognitive move. The individuals involved in this study clearly had moments when they shared ideas--listening closely to one another, and other moments when they ignored their partner's comments --retreating within their own heads to focus their attention on their own thoughts.

When an individual tunes out her partner, this filtering of infomation necessarily has an effect on the problem-solving process. Facts may be noted, yet promptly discarded; ideas may be suggested, yet inadequately explored; even correct solutions may be outlined, yet incorrectly implemented. Although it is difficult both to observe and to categorize, ignoring a partner is, in and of itself, an important metacognitive move.

On the Importance of Knowing *How* to Monitor

Metacognition has two essential components--awareness and regulation. In practice, this means that effective monitoring requires that one recognize not only what and when to monitor, but also--perhaps most importantly--how. It was striking how often the women in this study made comments that indicated they were aware that they should be planning, assessing, evaluating, verifying, or engaging in some other "monitoring" behavior. Yet, it was not at all rare for the women to fail to utilize such cues after they had been verbalized.

An implication of this observation is that it is clearly not enough to teach students that they should give themselves "cues" to plan ahead, to monitor progress, to check results, etc. The individuals in this study often did this spontaneously. However, these cues were too often faultily implemented, poorly attended to, or even totally ignored. Clearly it is necessary for students to be aware of what types of monitoring are useful, and when. But such knowledge is not sufficient. At least as important, and certainly more difficult, is knowing how to monitor. There are two parts to the definition of metacognition: awareness and regulation. Knowing what and when is only awareness; the essence of regulation is knowing how.

On the Negative Aspects of Persistence

The work of one of the pairs in the study was very like the persistent, but unreflective work that Schoenfeld (1985) found so typical of many of the college mathematics students he studied. Schoenfeld describes students who doggedly pursue a single unprofitable goal as on a "wild goose chase," and points out that most of his "students were remarkably consistent--and persistent--in pursuing wild mathematical geese" (p. 193). Schoenfeld further claims that problem-solving examples such as these are illustrative of the "disastrous consequences of an absence of self-regulation" (p. 193).

It is curious to note, however, that my system of identifying monitoring moves and events certainly did not turn up any particular absence of monitoring behaviors for Linda and Alice, the pair of students in my study who were the chief pursuers of "wild mathematical geese." Although they were clearly guilty of "an absence of self-regulation," they were nevertheless not lacking in monitoring moves. It was, instead, their reaction to their own monitoring moves that distinguished them from the other two pairs in the study. It seemed that even when they made monitoring statements that might have cued a less persistent individual to abandon an unprofitable effort and to move on to trying something else, Linda and Alice often tended to ignore their own observations and to persist with the same approach. Their protocols were nearly always the lengthiest by far.

Perhaps we can learn two things from Alice and Linda. It is not sufficient simply to make monitoring statements. It is how one reacts to such statments that makes a difference in the problem-solving process. And it is not necessarily a virtue to be persistent. A lost person who runs in the wrong direction for a long period of time usually ends up further from home than an individual who proceeds more slowly and cautiously, changing course whenever necessary as he frequently reevaluates his orientation. Persistence is not necessarily a virtue in problem solving.

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Effects of Extraneous Information and Multiple Steps on the Problem-Solving Performance of Middle School Students

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Abstract

To determine how middle school students cope with some of the demands imposed on them by nonroutine word problems, 140 eighth graders were asked to solve nonroutine word problems modeled after those used by the National Assessment of Educational Progress. Processing demands were imposed on the students by adding extraneous information and extra steps to the problems. Results indicated that the presence of extraneous information and extra steps reduced the accuracy of the students' solutions. In addition, the two measures of accuracy, correct answers and correct formulas, were positively correlated. The findings suggest that one way extraneous information and extra steps affect students' problem solving performance is by interfering with their ability to identify the problem question and the appropriate rule or formula for answering the question. Thinking-out-loud protocols also revealed several misconceptions that students have about solving word problems.

The purpose of the present study was to investigate why students hava difficulty solving nonroutine mathematics word problems. Systematic assessments conducted by the National Assessment of Educational Progress reveal that students perform poorly on nonroutine word problems (Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988). In the present study, nonroutine word problems were defined as (a) problems that contained extraneous information, and (b) problems that required more than one step to solve. These two types of nonroutine word problems are regularly included in NAEP assessments.

Nonroutine problems may be difficult for children to solve because they are susceptible to two classes of errors: (a) working memory errors, and (b) misconceptions. Working memory errors may occur when the presence of extraneous information and/or extra steps overload the limited processing capacity of an individual's working memory. The processing capacity of an individual is defined here as "the limited pool of energy, resources, or fuel, by which some cognitive operations or processes are mobilized and maintained" (Johnston & Heinz, 1978, p. 422). Because processing capacity is limited, the task of solving mathematics word problems can be difficult because an individual must extract quantitative information from a verbal context and manipulative that quantitative information in working memory (in accordance with recalled rules). When extraneous information and extra steps are added to a word problem, the difficulty of the task is compounded by extra requirements such as deciding which numbers must be used, how many steps are necessary, and in which order the steps should be performed.

Nonroutine word problems may also be susceptible to several misconceptions that students have about solving word problems (Muth, 1988a). For example, some student believe that all of the numbers in a word problem must be used. Other students believe that word problems must be solved in one step. And some students believe there is a relationship between the amount of quantitative information given in a problem and the number of steps required to solve it (e.g., if there are two numbers in a problem, it must be solved in one step; if there are three numbers in a problem, it must be solved in two steps, etc.). These three misconceptions could interfere with students' ability to solve problems with extraneous information and more than one step. However, at the present time, we have little information concerning how to go about changing these misconceptions, how large a role they play in students' ability to successfully solve nonroutine problems, or how these misconceptions might interact with working memory errors.

Despite the fact that we know something about why nonroutine problems are difficult to solve, very little is known about the exact nature of the higher-order skills required to solve them (Marshall, 1986). We know that students must integrate their reading comprehension and math computation skills (Muth, 1984, 1988b) in order to successfully solve nonroutine word problems, but the exact skills involved and how they interact is still not clear. If we expect classroom teachers and textbook authors to routinely include problems such as these in the curriculum, then we must supply them with better information about the exact nature of the skills involved in solving them and identify some of the errors students typically make when solving them. In order to begin to answer these questions, the following two-phase study was conducted.

PHASE 1

Method

Subjects and materials. The subjects were 140 eighth-grade students. A 14-item word problem test was constructed using adaptations of sample problems supplied by the National Assessment of Educational Progress. The problems delt with the following concepts: distance, interest, perimeter of a rectangle, area of a rectangle, circumference, area of a circle, and volume of a cylinder. Four versions of the test were formed by combining two variations of problem information (absence vs. presence of extraneous information) with two variations of number of steps (one vs. two). The following problem illustrates the four test versions:

No Extraneous Information - One Step

> Nancy exercises by jogging around a circular building. The building has a diameter of 100 yards. How far does Nancy jog every time she goes around the building?

Extraneous Information - One Step

> Nancy exercises by jogging around a circular building. The building is 75 yards high and has a diameter of 100 yards. How far does Nancy jog every time she goes around the building?

No Extraneous Information - Two Steps

> Nancy exercises by jogging around a circular building. The building has a radius of 50 yards. How far does Nancy jog every time she goes around the building?

Extraneous Information - Two Steps

> Nancy exercises by jogging around a circular building. The building is 75 yards high and has a radius of 50 yards. How far does Nancy jog every time she goes around the building?

Procedure. In six eighth-grade classes, the four versions of the test were randomly assigned to the students with the restriction that an equal number of students receive each version. The instructions encouraged the students to work carefully, reminded them to show all of their work, and instructed them to write down the correct formula for each problem even if they did not know how to solve the problems. (Even though these students were very familiar with the concepts involved in the problems, they were still required to learn the formula to criterion before they could participate in the study.)

Two measures of performance were used to assess the solution of the word problems: (a) total correct answers, and (b) total correct formulas. Students received one point for each problem that had been carried out correctly and whose final answer was correct. Students received one point for each problem that had a correct formula associated with it.

Results

Analyses of variance were performed to assess the effects of problem information and number of steps (between-subjects factors) on correct answers and correct formulas. Group means on these measures are shown in Table 1.

Table 1

Means for Performance Measures

	Problem Information		
Number of Steps	Extraneous Info Absent M	Extraneous Info Present M	
	Total Correct Answers		
One Step	7.11	4.23	(5.67)
Two Steps	2.60	1.80	(2.20)
M	(4.86)	(3.01)	
	Total Correct Formulas		
One Step	7.97	5.89	(6.93)
Two Steps	3.46	2.43	(2.95)
M	(5.71)	(4.16)	

Note: Means based on 35 subjects.

Results indicated that students correctly answered more problems when extraneous information was absent than when extraneous information was present and when the problems required one step rather than two steps to solve. The preceding main effects were qualified by a Problem Information X Number of Steps interaction which suggests that the presence of extraneous information exerted its primary influence on one-step problems. The interaction may be due to a "floor" effect on two-step problems. Regardless of the presence or absence of extraneous information, students solved very few of the two-step problems correctly.

Results indicated that students listed more correct formulas when extraneous information was absent than when extraneous information was present and when the problems required one step rather than two steps to solve.

Correct answers was correlated positively with correct formulas ($r = .95$, $p < .001$).

Discussion

The presence of extraneous information and extra steps in word problems reduced the accuracy of students' answers. This finding is consistent with the hypothesis that the presence of extraneous information and extra steps imposed formidable demands on students' limited processing capacities. In addition, the high correlation between correct answers and correct formulas suggests that students experienced most of their difficulty trying to determine what the problem was asking and then recalling the appropriate formula. Those students who started off a problem with the correct formula experienced very little difficulty solving the problem correctly. Perhaps they were able to use the formula as a guide to determine which information to use and how many steps to perform.

PHASE 2

Method

In order to explore some of the misconceptions associated with solving nonroutine word problems, eight students (none of whom had participated in the first study) were randomly selected from six eighth grade classes to think outloud as they solved nonroutine word problems. Larkin and Rainard's (1984) and Muth, Glynn, Britton and Grave's (in press) suggestions for collecting think-aloud data were followed. The transcript of one student, Jennifer, attempting to solve the extraneous information - two step circumference problem described previously is presented below.

Extraneous Information - Two Steps
(Jennifer)

(Reads problem out loud.)
(Doesn't say anything for about 8 seconds.)
Teacher: What are you thinking right now?
Jennifer: um, that I multiply, um, the radius, which is, um, and I square, um, 50 so it'd be 50 times itself, and I get 2500 times . . . 75
(Works for a long time on the multiplication.)
Teacher: So do you have an answer?
Jennifer: um, . . . 197,500
(Long pause)
Teacher: What are you thinking right now?
(Long pause)
Jennifer: It doesn't really sound right.
Teacher: Why does it not sound right?
Jennifer: Sounds too big. The number's too big.
Teacher: So what are you thinking?
(Long pause)
Teacher: You think that it's not right but you're not sure what to do? Is that what it is?
Jennifer: Yea
Teacher: So you want to leave it at that?
Jennifer: Yes m'am, I don't know what it is.
Teacher: ok

Discussion

Jennifer's transcript is representative of the transcripts of the other students who solved this problem outloud. (Only one student solved it correctly.) The transcript suggests that Jennifer has the misconception that she should use all of the numbers in the problem. Her answer turns out very large, she realizes it's too large, but doesn't know what to do about it. Also evident from the transcript is the fact that Jennifer never identifies the problem as a circumference problem or writes down the formula for circumference. Instead, it appears that she examines the numbers given in the problem and decides which formula the numbers go with. That is, Jennifer sees that the radius is given and probably recalls a formula where radius is used (i.e., area = πr^2) and proceeds from there. Unfortunately, she gets lost along the way.

This preliminary work using thinking-out-loud protocols suggests that some students have misconceptions about solving word problems and these misconceptions affect their ability to solve nonroutine word problems. The thinking-out-loud protocols will be discussed in more detail in the oral presentation.

GENERAL DISCUSSION

Students should be exposed to nonroutine word problems such as those containing extraneous information and extra steps because they are a good test of the students' understanding of the concept(s) involved in the problems and because they are typical of many of the problems they will eventually encounter and be expected to solve in real life. However, as the results of the present study indicate problems such as these are difficult for students to solve because they are susceptible to working memory errors and misconceptions. Before recommendations can be made about curricula design additional research is needed to identify the exact nature of the processes necessary for solving nonroutine problems and to identify those processes that are potentially troublesome for students.

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PRESERVICE ELEMENTARY TEACHERS' CRITICAL THINKING SKILLS AND HEURISTICAL PROBLEM SOLVING ABILITY

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The Watson-Glaser Critical Thinking Appraisal, M.A.A. Advance Algebra Exam, and a 7-item, open-ended problem solving instrument were administered to 74 preservice elementary teachers. An analysis of the students' strategies used on individual problem solving tasks indicated a possible relationship between critical thinking and strategy selection.

Results from recent national and international studies (e.g. Brown, et al, 1988; McKnight, et al, 1987) have fueled a renewed interest in higher order thinking skills among American educators. However, to date, there is no consensus as to a definition of higher order thinking. As noted by Resnick (1987),

> Many candidate definitons are available. Philosophers promote critical thinking and logical reasoning skills, developmental psychologists point to metacogniton, and cognitive scientists study cognitive stategies and problem solving...How are they related to the problem solving abilities that mathematicians, scientists, and engineers try to teach their students? (p. 1)

Recently, several researchers (e.g. Blai, 1986; Preisseisen, 1986) have proposed that higher order thinking can be subdivided into four categories: Problem solving, Decision making, Critical thinking, and Creative thinking. It is not necessarily true, however, that each component of this classification holds equal weight. Indeed, Marzano and colleagues (1988) believe that "critical thinking and creative thinking should not be considered processes comparable to problem solving and decision making [since] these terms imply

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judgment about the quality of the thinking involved" (p. 17).

Although Marzano outlined the nature of the interrelationships among these four categories of thinking processes, research is needed to verify these proposed connections. Thus, the intent of this study is to explore the link between two of the categories; namely, critical thinking and mathematical problem solving. Specifically, the objectives are as follows:

1. To analyze the critical thinking ability of elementary education majors.
2. To explore the relationships between these students' critical thinking skills and heuristical problem solving ability.
3. To analyze these students' approaches to various problem solving tasks.
4. To analyze the relationships among these students' understanding of problems, solution strategies used, and final answers on open-ended problem solving tasks.

METHODS

Subjects and Procedures

During Fall 1987, the 178 elementary education majors enrolled in 4 sections of the mathematics content course required for certification received instruction in problem solving strategies. A text by Merserve, Sobel, and Dossey (1987) was used in this course and served as the basis for the problem solving instruction.

An algebra compentence exam was administered to these classes at the beginning of the fall semester. Near the end of the semester, after the completion of the problem solving instruction, 74 of these students volunteered to complete two additional instruments; one assessing critical thinking and the other problem solving. An approximately equal number of students volunteered from each of the four course sections. Of the 74 volunteers, 46% were first year students and all except one were females.

Instruments

The algebra compentence measurement was a 25-item multiple choice advanced algebra exam developed by the Mathematical Association of America (1983).

To assess critical thinking, the Watson-Glaser Critical Thinking Appraisal (Watson and Glaser, 1980) was used. It is an 80-item multiple choice test which measures five thinking processes associated with critical thinking: Inference, Recognition of assumptions, Deduction, Interpretaton, and Evaluation of arguments.

The problem solving instrument contained seven open-ended items, such as the Mangoes problem stated in Figure 1, selected from several problem solving studies (e.g. Oprea, Sutton, and Fleming, 1987; Stonewater and Kullman, 1985) and represented a diversity of possible problem solving heuristics. A scoring method, devised by Charles, Lester, and O'Daffer (1987), was used in which problems are rated 0 to 2 on each of three subprocesses: Understanding the problem, Planning a solution, and Getting an answer. Independent rating by two raters had an interrater reliability of 84%.

Of a collection of mango fruit, the king took 1/6, the queen 1/5 of the remaining mangoes and the three chief princes 1/4, 1/3, and 1/2 of the successive reminders. Then the youngest royal child took the remaining three mangoes. Oh you who are clever in miscellaneous problems on fractions, give out the number of the original collection of mangoes.

Figure 1. The Mangoes problem from Problem Solving instrument.

RESULTS

The following are the overall means for the three instruments: Algbra (mean = 11.19, possible = 25, n = 178), Problem Solving (mean = 26.8, possible = 42, n = 74), and Critical Thinking (mean = 46.8, possible = 80, n = 74). The Critical Thinking subscale scores were as follows: Interpretation (mean = 10.5), Assumptions (mean = 9.5), Deduction (mean = 9.6), Evaluation (mean = 8.8), and Inference (mean = 8.4). As indicated in Table 1, the students varied considerably in their ability to solve the different problems with mean scores ranging from 5.4 to 1.5.

Table One
Problem Solving Means by Problem and Subprocess

Problem	Total	Understanding	Strategy	Answer
1. Tennis	5.2	1.8	1.8	1.6
2. Darts	4.8	1.6	1.7	1.5
3. Jobs	5.4	1.8	1.8	1.8
4. Mt. Climb	4.9	1.7	1.7	1.5
5. Exponent	1.5	0.7	0.5	0.3
6. Mangoes	3.4	1.3	1.2	0.9
7. Rectangles	1.6	0.8	0.5	0.3
Overall	26.8	9.7	9.1	7.9

For each of the 7 Problem Solving items, the overall and 3 subscale scores were correlated with the Algebra score and the 5 Critical Thinking subscores. These correlations were all <= .3 and were not considered statistically significant.

When possible, the strategy a student used on each individual problem solving item was identified. The students were then categorized according to the type of strategy used and whether their solution was correct. The mean Critical Thinking and Algebra scores were calculated for the students on these various categories. Table 2 lists the various identifiable strategies used for the Mangoes problem and the mean Critical Thinking and Algebra scores of the students who used these strategies.

Table Two
Critical Thinking and Algebra Scores according to the
Strategy used on the Mangoes Problem Solving Task

Strategy used	Solution[1]	N	Crit. Thinking	Algebra
Work Backwards	correct	20	49.90	12.95
	wrong	21	47.57	11.24
Algebra	correct	4	45.50	11.25
	wrong	9	46.77	9.56
Diagram	correct	3	50.33	12.67
Guess/Check	correct	2	56.50	13.50
Add Fractions	wrong	2	53.50	21.00
Other	correct	4	58.00	11.75
	wrong	9	36.11	11.00

[1] Indicates students with correct or incorrect solutions.

The students used a variety of strategies on this problem with a majority (73%) attempting to solve it by working backwards or writing an algebraic equation. The students who attempted the Algebra strategy had the poorest algebra scores (mean = 10.1, n = 13). In general, regardless of the strategy used (except Add Fractions), the students with an incorrect solution had a lower Algebra mean score then the students using the same strategy that had a correct solution. Similarily, except for Add Fractions and Algebra strategy categories, the students with an incorrect solution had a lower Critical Thinking mean score. The students in the Other category had the lowest Critical Thinking mean score (mean = 42.85, n = 13) while the ones in the Guess/Check category had

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the highest (mean = 56.60, n = 2).

CONCLUSIONS

The intent of this study is to explore the link between critical thinking skills and mathematical problem solving ability of elementary education majors. While the correlations among problem solving, algebra, and critical thinking scores were not statistically significant, the analysis of the students' critical thinking and algebra skills according to the strategies they used on various problem solving tasks did indicate a possible relationship between critical thinking and strategy selection. However, these findings must be tempered with the small number of students that selected some of the heuristics (e.g. Only 2 students used Guess/Check on the Mangoes problem). Additional data should be collected to assess the validity of this conjecture. Furthermore, individual interviews and direct observation are necessary to reveal the full range and depth of these students' problem solving ability and the complexity of their critical thinking processes.

According to Schoenfeld (1983), "purely cognitive behavior is extremely rare, and that what is often taken for pure cognition is actually shaped - if not distorted - by a variety of factors" (p. 330). He proposes a matrix, within which pure cognition resides, that consists of environmental factors, beliefs, and metacognition. Future research assessing elementary education majors' beliefs and metacognitive strategies may further clarify the relationship between the cognitive processes of critical thinking and problem solving.

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Multiplication and Splitting: Their Role in Understanding Exponential Functions[1]

Jere Confrey, Cornell University[2]

An understanding of exponential functions requires an understanding of repeated multiplication. The paper makes the conjecture that the concept of multipliciation as repeated addition is not adequate for repeated multiplication and that a partitive concept of multiplication through splitting (as in reproduction) is more appropriate.

Exponential functions are useful in modelling hundreds of phenomena in such diverse fields as economics, information theory, particle physics, biology and engineering. However, their introduction in the secondary curriculum, is cursory,with applications often reduced to substituting into formulae. In this project, we seek to establish what it might mean for students to understand the concept of exponential functions so as to be able to recognize appropriate settings for their use as a modelling tool and to do so across a variety of representations; numeric, algebraic, and graphical (trees, embedded squares, number lines, Cartesian plane,etc.).

To study exponential functions, we are conducting a series of historical and interdisciplinary inquiries into the multiple roots of exponential growth. These studies and our experiences in the classroom have assisted us in the design of teaching interviews with college-level students on exponential functions. The assertions and conjecture in this paper resulted from a detailed analysis of one of these interviews with a plant science major who was interviewed for nine one-hour interviews over the course of four weeks. Four months later, he returned and participated in an additional four hours of interviews over four weeks. A more complete report of his case and methodology is available (Confrey, 1988).

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[1] This work was supported under a grant from the National Science Foundation No. MDR8652160. All opinions and findings are those of the author and not necessarily representative of the sponsoring agency.

[2] Although the identification of an independent splitting structure originated with the author, the development of these ideas came through extensive discussions of the project staff and colleagues: Erick Smith, Julian Fleron, Jan Rizzuti, David Henderson, Wendy Millroy and Kenneth Carlstrom.

In this brief paper, I wish to make three assertions about what it means to understand functions and then one rather bold, perhaps premature, conjecture about the operation of multiplication as splitting. The majority of the paper will be devoted to discussing the conjecture. At the time of the conference, a longer version of this paper will be available with excerpts from the cases demonstrating how one gains explanatory power with such a conjecture.

A Theoretical Framework for Exponential Functions

We began with the expectation of spending a brief amount of time with the students reviewing the rules of exponents and their application in scientific notation and then moving quickly to explore their understanding of exponential functions. To our surprise, the understanding of exponential expressions and the rules of operation were far more problematic than we had anticipated and led to a rather in-depth pursuit of their understanding of the operation of multiplication, and in particular, of repeated multiplication.

Three Assertions

A1. Students' understanding of exponential functions needs to be well-grounded in an understanding of the operational character of functions. All too often, students perceive the "general form" of an equation as a string of symbols whose positional, alphabetic character overrides their perception of the underlying operations. Thus, exponents are seen as symbols whose placement (and manipulation of that placement) on the page is prominent. (For example, x^{-3} is equal to one over x^3: a transformation in position.) The connections to repeated multiplication and ratio are obscure. Thus, we

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claim that the power of the concept of functions is diminished by the absence of emphasis on the operational character of classes of functions.

A2. Successful application of exponential functions to model phenomena requires that students' recognize the actions which correspond to the operational character of the function. Thus, if multiplication is understood to correspond with splitting or magnifying, then students will seek to model phenomena in which this action is repeated with exponential functions. Thus, we assert that through the increased emphasis on operations and the actions which correspond to them, students will demonstrate improved performance on contextual problems.

A3. Attention to this operational character of functions may lead to revision in the presentation of variables and functions in order to focus more on difference and change in the dependent variable ($y_1 \rightarrow y_2$) rather than on the input-output (x produces y) perspective of our traditional f(x) notation. Thus, we claim that in creating and exploring multiple representations of functions, we need to pay close attention to the ease with which the change in the dependent variable as the independent variable varies is coded in those representations.

Since all of these claims about functions are dependent on a student developing a deep understanding of operations, we found it necessary to explore the concept of multiplication in more depth and, in particular, the concept of repeated multiplication. Furthermore, in trying to understand the concept of repeated multiplication, we found it helpful to view the exponent as a counter (a conception limited to the positive integers). Viewing it as an integer led us to a careful examination of the isomorphism between the addition and multiplication of exponents and the multiplication and root/powers of exponential expressions. In doing so, we found evidence of two different conceptions of multiplication on the different sides of the isomorphism and that led us to make the following conjecture:

C1. There is a concept of multiplication which is not adequately described by the idea of repeated addition nor the ratio concept. This concept of multiplication is connected to the actions of splitting and of magnifying (or shrinking) and starts with the unit (one) rather than the arithmetic origin (zero). A child's experience of the actions of splitting and magnifying develops relatively independently of his/her experience with counting or enumerating, although a mapping does occur from one to the other. Our ways of teaching multiplication fail to distinguish multiplication as repeated addition from multiplication as splitting or enlarging and, as a result, the additive/counting structures become confused with this multiplicative/divisional construct. We need to strengthen students' understanding of this alternative view of multiplication and then carefully build the bridge among constructs of multiplication through activities designed to explore their isomorphic relationships.

In the next sections I will discuss the conjecture C1 in more detail, relating it to the literature on multiplication and division and to excerpts in the case study.

The Conjecture

Fischbein and his colleagues (Fischbein, 1983; Fischbein, Deri, Nell and Marino, 1985) allege that: " Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model. Identification of the operation needed to solve a problem with two items of numerical data takes place not directly but is mediated by the model. The model imposes its own constraints on the search process." (1985, p. 4.) They further explain that 1) like Piaget (1970) they assume that operations are linked to practical behaviors--"the performable counterpart of the operation" (Fischbein et al. p.5) and 2) that changes in the formal status of the operation may occur without corresponding transformations in the primitive models--thus they are enduring. Furthermore, they are careful to claim no innate status for

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these primitive models: they are derived from experience and school experience is likely to be an influential source on this process.

We agree with Fischbein et al. that such primitive models are linked with the operations and that the roots of the operations lie in the behaviorial actions connected with multiplication. We further believe that operations are formalizations of mental actions into systems of representations of actions. [Piaget (1970) calls them internalized mental actions.] We do not believe these are necessarily unconscious, only that they are not well- differentiated because of our lack of reflection on them both individually and in our forms of schooling. Thus, we continue to use the terms: primitive models, frames, schemes or mental structures to identify forms of representations of actions which are seen to be similar and which we find repeatedly useful in our everyday activities. As stated simply by Sinclair (1987), this form of constructivism leads us to "see the construction of mathematical objects and operations as a slow construction, deeply rooted in all human endeavors to make sense of their world."(p35).

Fischbein et. al continue positing a set of such primitive models for the four basic operations, and these models find support in the work of Greer and Mangan (1984) and Bell, Fischbein and Greer (1984). For addition, Fischbein and colleagues suggest "putting together" and for subtraction, they suggest "take away" and "building up". For multiplication, they posit that repeated addition constitutes the prevalent primitive model wherein there are two entities, an operator (multiplier) and an operand(multiplicand) and the operand is added the number of times of the operator. Empirically, they found support that when the operator (the n in mxn) is non-integral, the students of all ages drop in performance; a finding whose impact is mediated by what they term an"absorption effect" whereby for larger decimal values (such as 3.25 as compared to .75 or 1.25), they assumed the decimal portion is absorbed, easing the difficulty.

For division, they suggest two models: partitive and quotitive division. Briefly, partitive division interprets $m/ n= p$ as m things divided into n piles of p objects. Quotitive division is likened to repeated subtraction, thus answering the question "how many groups of size n can one take out of m?" In their research, they found more support for the presence of the partitive than the quotitive model, especially prior to grade nine.

Steffe (1988 a and b) takes issue with the simplicity of the quotitive model providing a more precise analysis of what a child must learn to do to have developed the quotitive model including the construction of the unit of size n as a composite unit rather than as a method of segmenting a quantiy. This emphasis on the role of the unit in understanding multiplication is fundamental in our reconceptualization of multiplication.

Thus, what emerges from this set of researchers is a model which appears surprisingly asymmetric and which, in most cases, assumes a clear distinction between the operations of division and multiplication. We believe it can be argued that in this model, quotitive division (repeated subtraction) can be closely associated with multiplication as repeated addition which leaves no multiplicative interpretation of partitive division. See Figure 1.

Figure 1

Multiplication	Division
....	partitive
repeated addition	quotitive (repeated subtraction)

We wish to alleviate this asymmetry by proposing the following: Since multiplication is the reversal of division, it will necessarily have corresponding structures. In fact, to treat multiplication and division as independent constructions is unwise. Thus, we expected that there could be a related multiplicative companion to partitive division. We have identified and named such a concept: splitting. Splitting

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frequently is identified with division, however, in this concept of splitting, as in reproduction, the entity is reproduced through a splitting apart action.

Support for this approach comes from Piaget's four characteristics of operations (1970):

1. The first is that operations are internalized mental action, which we can seek in human activity. The actions of splitting, tearing apart and equal sharing develop at an early age in settings distinguishable from counting (often measurement based).

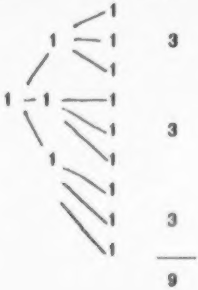
2. The second characteristic is that of reversibility (also see Krutetskii, 1976). This emphasis on reversibility can be taken to suggest that multiplication as the reversal of division ought to involve a corresponding relationship to the action of partitioning. That the splitting construct seems to confuse division and multiplication is not of great concern, for in a sense, when one divides a cake into four pieces, both four and a fourth have been created. The distinction depends on whether one reunitizes immediately (obtaining four) or maintains the previous unit for one round (obtaining a fourth).

3. Piaget's third characteristic is that an operation supposes some invariant or conservation. Vergnaud's work (1979,1983a,1983b) on multiplicative structures which posits "an isomorphism of measures" seems to be uniquely concerned with the invariances among ratio, division, multiplicative and linear concepts. In the splitting construct, the invariant quality may lie in how the split is created. Halving or doubling a quantity may be a primitive, but one cannot presume it is accomplished by counting out into two piles. It could also be achieved by a cut or a folding of a continuous quantity in a measurement setting. Within such a perspective, one would expect that fourths, eighths, etc. as splits of splits would be more accessible than say, sevenths or thirteenths. Thus, within the splitting conceptions, one would expect numbers with multiple factors to be most accessible.

4. Finally, Piaget suggests that an operation exists within a structure. To describe the structure in which this conception of multiplication resides, we start with splitting as the basic action; one splits (reproduces) one into K copies (not necessarily by stamping them out, but as with cells by splitting).

In order to establish such a system, one can create a progression using Peano's axioms of 1, n^2, n^3 as the result of repeated splits of n (n is a unitary operator). Just as in Peano zero is undefined; in this system, one is undefined. Multiplication can be defined among the terms in the sequence such that n^k / n^m = k n-splits followed by m n-splits. Multiplication of non-equivalent bases must be defined and we will report on this work in the longer paper. Since repeated multiplication is our concern, this is sufficient for exponential functions.

Most people, when they first encounter this claim will quickly object that splitting is repeated addition. Their argument goes like this: three times three gives three groups of three and if one adds up these three threes, one gets nine. This claim cannot be disputed; the result of such an action is named nine. But this misses the point and the tree diagram below may be useful in discussing the difference.



We claim that the action of splitting does not need to be totalled to have generated the necessary successor. One needs only to know how to create a split and to have a way to be sure there are equal amounts in each pile to have completed the action. That quantity produced by the split could have been independently named and learned just as the familar "number sequence" is learned in counting. The independence of the splitting structure from counting is assured as long as we can imagine the action of splitting a quantity into two by symmetry, balance or folding; some method that does not rely on counting.

The fact that we wisely choose to perform a counting act on the outcome and to name the total by the same name as we have in counting is an isomorphic act, creating a correspondence between these two structures, splitting and counting. In the counting structures, multiplication may be considered as repeated addition (at least for integral values)[1], but to portray multiplication as solely repeated addition dismisses the actions inherent in creating this specific structure. In the action of splitting, the groups are formed simultaneously and the image is one of nested rather than consecutive sets. To prematurely reduce this to repeated addition within the counting structure is to miss these essential elements.

The definition of multiplication suggested by Piaget and Szeminska (1941/1967) allows for a splitting interpretation of multiplication wherein the equivalence of the sets is assured by a form of one-to-one correspondence. There is no indication of the necessity of enumerating in it.: "An arithmetical multiplication is an equi-distribution: m x n is m set of n terms or n sets of m terms which correspond one to one. "(Quoted from Hersocovics, 1983, p. 194.)

Thus we make the conjecture that splitting is an appropriate action for modelling partitive multiplication, that this structure is relatively independent of the counting generated structures of traditional arithmetic. However, we need to conduct the same kind of careful analysis as is available on the counting types (Steffe, von Glasersfeld, Richards and Cobb, 1983) and addition and subtraction (Carpenter and Moser, 1983). We further conjecture that since the product of a split can be described isomorphically to repeated addition, that students' awareness of this structure is low and it is frequently prematurely assimilated into a definition for multiplication as repeated addition.

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[1] In this paper, we only discuss the integral values for the structures described. In future work, we plan to work further on the extension into rationals and reals as is essential to move from the splitting construct to that of magnification.

At this stage, we wish to stress how tentative this conjecture is. Its value lies in its potential to help us sort out certain episodes that recur in our data. Students show a very quick tendency to reason on problems involving multiplicative splitting, suggesting its intuitive appeal in isolation. They also show evidence of confusion about whether a process originates with zero or one, and specifying an origin of one for splitting clarifies this confusion in our analysis. Finally, they show disturbing tendencies to obsure the distinctions between multiplicative and additive relations. We suspect that creating microworlds where they gain the familiarity in operating more formally with splitting structures will facilitate an extension into non-integral multipliers such as are involved in magnification. Finally, we hope that through such a process, we can assist students in gaining a deeper understanding of the circumstances in which modelling by exponential growth would be productive.

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A CONSTRUCTION OF THE CONCEPT OF EXPONENTIAL FUNCTIONS

Jan Rizzuti and Jere Confrey, Cornell University[1]

This selection from a larger case study focuses on a student's construction and interpretation of an exponential function. The contextual nature of the problem and the student's ability to see and represent change play important roles in the conceptualization.

The purpose of this paper is to share rich, descriptive data which support our developing understanding of how students conceptualize exponential functions. The case study was developed from video tapes and verbatim transcriptions of nine hours of teaching interviews conducted over the course of four weeks with "Sally", a sophomore nutritional sciences major.

In the first four interviews Sally explores the meanings of positive, negative, and fractional exponents as she simplifies exponential algebraic expressions, represents repeated multiplication pictorially, and calculates with numbers expressed in scientific notation. In the last five interviews she develops a sense of the characteristics of exponential functions as distinct from those of linear functions as she grapples with several contextual problems involving depreciation, compound interest, binary searches, and linear decay. Initially she models all growth and decay as linear functions. She expects additive rates of change and linear graphs. The following brief selections chart her course of development through her first efforts to construct an exponential function and highlight the difficulty she has in developing an understanding of the effects of multiplicative rates of change.

In the fifth interview Sally is given a depreciation problem involving a computer valued at $4000 which loses 10% of its value each year, and she is asked to find its value after one year. She calculates the amount of loss and the resulting value of the computer by multiplying 4000 x .1 and subtracting the loss from 4000. The next part of the problem asks for the value after three years, and she predicts that the amount of loss would be triple the amount lost in the first year. She immediately raises the question of whether or not she is interpreting the problem correctly. "After three years ... it would be 400 times 3. The original loss after one year, or no. You'd take the new

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[1] This work was supported by the National Science Foundation grant number MDR 8652160 to the second author.

value. I don't know if they mean the original worth or ...I'm not sure if they mean 10% of each, what it's worth before or the original." [Day 5/Transcription page 19]. The interviewer explains that it loses 10% of the value it has each year, and Sally calculates the value of the computer after the second year by multiplying her previous answer by .1 and subtracting it from 3600. She then calculates the value after the third year in the same manner. She stops when the next problem asks for the value after ten years. "There must be an easier way. I recognize a pattern, but I don't really know it. So I'd have to go through it ten times." [5/20].

She and the interviewer spend several minutes working on a way to represent the amount present after each year's loss in order to come up with a way to find the value after 10 years. She writes 4000 - (4000 x.1), draws a box around it and refers to it as "BOX". For subsequent years she writes BOX - BOX(.1), which she calls BOX', and BOX' -BOX'(.1). She does not find these generalizations helpful: "The only problem is that you have a new value each time for BOX." [5/25]. She sees that multiplying by .1 moves the decimal point over one place and rewrites .1 as 10^{-1}. Looking at BOX - (BOX x10^{-1}) she decides to "assign variables" [5/26] to the expression and writes B - (B x 10^{-1}). The interviewer offers A-AX as a comparable algebraic expression, and Sally sees that factoring may be helpful. She factors variable B out of the expression rewriting it as B(1-.1), and she sees that B x .9 is another way of representing this.

However, when asked to interpret this in terms of the depreciation problem, she does not see that B x .9 is accomplishing both the multiplication and subtraction steps of B - (B x 10^{-1}). She cannot construct a meaning for the .9. Shifting to an easier problem the interviewer asks for the value of $100 after a 10% loss and Sally says "$90. I just subtracted [10% of 100 from 100]." [5/29]. The interviewer then helps her see that losing 10% is the same as keeping 90%, although Sally at first calls it "gaining 90%". [5/29]. She now sees that multiplying by .9 is another way to calculate the value after one year. She says it makes sense and she declares that this procedure is "a lot easier than subtracting it." [5/30]. Throughout the nine interviews Sally relies to a great extent on her actions with the operations involved in problems, and in this exchange she has difficulty seeing two operations replaced by one. The generalizing and factoring are not as useful as the numerical example in her final realization.

Having made sense of the .9 they continue to construct the function. The interviewer asks how much she would have at the end of the second year if she has B x .9 at the end of the first year, and Sally says "It would be "90% of 90%. .9 times .9 would equal 81% of the original .9." [5/30]. What looks like an insight into the exponential nature of this function may just be a lucky guess. She does not seem to know that the

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81% should be of the original value. And she does not apply this conjecture when she immediately guesses that to calculate the value after ten years she would "multiply 10 by [4000 x .9]." [5/31]. The interviewer suggests trying it on a value they know, but when Sally multiplies 4000 x .9 on a calculator the calculator does not work properly, and she leaves the table to get a different one before continuing. This distraction may have caused her to forget the conjecture altogether. As they proceed the interviewer says "Now, starting with $4000, how would you find out how much it would be worth at the end of two years?" [5/32]. Sally says "I'd multiply, multiply this, multiply the .9 by 2. Let's see. That might work. (On a calculator she gets 1.8 which she multiplies by 4000. The answer is not the expected value.) That might not work." [5/32-33]. At this point the interviewer asks her what she got when she multiplied 4000 by .9 and what she has to do with that answer to calculate the amount of money at the end of the next year. She says that she would "multiply [the 3600] again by .9" [5/33]. The interviewer reminds her that they do not want to figure out each successive year's value, and Sally says "Maybe it would be (pause) like .9 minus 1.8." [5/34]. The interviewer directs her attention back to the value of 3600, what to do to it next and where it came from. The way Sally records these transactions helps her to finally see the exponential nature of this function. She writes 3600 x .9 and then above the 3600 she writes 4000 x .9. [See Figure 1.]

4000 x .9

Figure 1 3600 x .9

She notices the cumulative nature of the repeated multiplication immediately, saying "Oh, I think I know where the twos and threes come in. By the .9s? It wouldn't be multiplied by two. It would be multiplied by itself each time. That would be $.9^2$." [5/34]. She multiplies 4000 x $(.9)^2$ on a calculator, and, getting the expected value, she says "That's where the two comes from. For three years it would be 4000 times .9 cubed. I'm going to check it out. (She works it out on a calculator.) It came out, that came out right, too!" [5/35]. Her confidence in this procedure was tentative after the first check, but after the second verification she seems to trust her procedure, and she confidently finds of the value of the computer after ten years.

As this session ends Sally is asked to explain why 4000 x $.9^{10}$ is the correct expression to calculate the value after ten years. She explains, "Well, originally they wanted what the value of 10% of 4000 was. And I multiplied .1 times the amount it was worth originally, and that gave me just 10% and it took the long way. And I subtracted the original value, um, took the original value minus, um, the percent loss and that would give me the answer. But I had, I had trouble replicating it again. I found a new value for "BOX" each time and I really didn't have to. It was, I think it was the original, the original 4000 and multiply it by 90% which is what I have left. And depending on how many times, how many years went by, I just raised it to the power of the year. Because you're taking 90% of each consecutive year, because that's what you have." [5/36-37]. This explanation focuses on what she perceives as incorrect actions on her part, rather than on why the function involves a base of .9 raised to the tenth power. She does not clearly explain why 90% is what remains and why exponentiation is involved.

At the beginning of the next session one week later Sally recalls the use of exponentiation and its relationship to the number of years in the depreciation problem from the previous session. "I think we were doing depreciation of value, interest. We squared or cubed, depending on the amount, what the year is. Say it was two years, then this is squared. Three years is cubed." [6/1]. Asked to rework the first parts of the problem she immediately says "I don't think I remember the formula." [6/1]. She laughs nervously as she says this. She does not seem to have a sense of being able to reconstruct the work from the previous week. She believes if she has not made a conscious effort to remember it, then it is gone. She begins the problem saying, "Before, when I originally did it I multiplied, um, 4000 by 10%. In figuring out the rest of the whole problem after ten years, it was time-consuming and I think we maybe decided to multiply by 90%? Because that's what we would have in the future? And (pause) so I'm going to multiply 4000 times 90%. And we want to know how much it would be worth after three years. And I think we could start with the original value, 4000, and multiply it by (pause) let's see, .90 cubed." [6/1-2]. The interviewer then asks her if that seems to be right, and she says "It makes sense. I could go through it the long way. I'm pretty sure it's [right]." [6/2]. Asked to explain why 4000 x $.90^3$ should work she says, "Well, this is the percent. You're always going to have 90% left of the original cost. So it's going to be 90%. After, after one year it's 3600. You take 90% of 3600. [It] would be somewhere in between 2916 and 4000. And there's - you multiply it by, I mean, raise it to the exponent by how many years it is." [6/2-3]. When the interviewer asks "What does the exponent do?", she says "It just keeps track of the years. How many years." [6/3]. But she is unable to explain why. When pushed to explain the exponent she resorts back to her original procedure. "I don't know. The long way would be to take a new value and do it all over again." [6/5].

Her record of the transactions today [See Figure 2.] does not help her see exponentiation as did the representations in the previous session, even though the iterative quality of exponentiation is highlighted.

Figure 2

$$4000 \times .90 = 3600$$
$$3600 \times .90 = 3240$$
$$3240 \times .90 = 2916$$

Looking at this representation she sees that ".90 is common to every equation. And .90 times .90 times .90 equals .90 cubed. Instead of writing out the difference each time you can just write 4000, which is what it would be worth, the original worth of the computer. (Pause.) And this is .90 cubed. That would be the worth after three years." [6/6]. She seems to be searching for any reason that may satisfy the interviewer's queries of "Why?".

Sensing a jump in her reasoning the interviewer asks Sally to explain why the value after three years involves .90 "cubed rather than three times .90 or repeated subtraction or something. ...Can you ...figure out a way to see that exponent in the problem?" [6/6-7]. Sally looks at the formula for a long time and replies "I don't think so." [6/7]. Her admission that she cannot create an explanation is consistent with her behavior in other interviews. She is reluctant to take risks and persists only as long as there are obvious possible explanations. In none of the nine hours does she question "Why?" or actively seek reasons for mathematical concepts.

Hoping to guide her to discover the exponentiation the interviewer poses a simpler problem, asking her to re-examine the various procedures she can use to calculate the value at the end of the second year. Sally states "4000 times $.90^2$. [And the long way] would be 4000 minus 4000 times 10%." [6/7]. Asked for another way she says "4000 times .90 and you would get a value and then that value times .90." [6/7]. The interviewer asks her what she would do if she could not use the equals key on the calculator, and she works out and writes 4000 x .9 x .9 saying, "[I]t's just the same as saying it's $(.90)^2$. ...I see. You don't have to get a new value each time." [6/8]. This statement seems to touch upon an important aspect of this problem. Finding subsequent values without actually writing down the intermediate values is very important in her final realization, but it is not obvious to her. Finally, through careful questioning the exponentiation is made explicit.

She finishes this part of the problem, finding the value of the computer after ten years using 4000 x $.90^{10}$, and she moves on. "The next question is: Does it become completely worthless and when? I don't know the answer to that. I guess you would try numbers until you come out with zero. You would [come out with zero eventually]." [6/9]. The interviewer asks her to guess when the value will be zero. "Well, I, when you start with \$4000, after ten years it was less than half of 4000, \$1394, so probably between 15 and 20 years. ...I guess I'll try 15." [6/9]. She calculates 4000 x $.90^{15}$, getting \$823.56, and 4000 x $.90^{20}$, getting \$486.31, and makes an observation. "Looks like it's being halved. That makes sense, right? [It's being halved] from 15 to 20 [years]." [6/10]. (The actual multiplicative rate of change between any span of five years, $.9^5$, is .59 so her observation is quite close.) She is able to see the changes from one value of a function to the next, but her expectations of the function do not seem to depend upon whether they are multiplicative changes or additive changes.

She predicts that the value will be halved again at 25 years and verifies this on a calculator. To quicken the pace she next tries 50 years, guessing that the value will be close to zero. She records the value of \$20.62 and then begins to approximate in her head. "I'm trying to figure. 55 will be ten, 60 will be five, 65 will be two. Maybe it will never be - I don't know. It's half each time. I could try 75. That's \$1.48. Almost \$1.50. [For 80 years it would be] 75 cents. It's 88 [cents]." [6/11]. She sees that her successive approximations are not approaching zero very quickly so she says "I'm going to try 200 or something and see if it works out." [6/11]. Looking at 4000 x $.90^{200}$ = .00000282 she says "That's pretty close to zero." [6/11]. In spite of this calculation she seems to expect the value to become zero and she asks the interviewer if she needs "to find the exact value" [6/13] when the computer is worthless. The interviewer says no and they move on to the next part of the problem.

Her prediction of the graph of this function ($4000 \times (.90)^T = V$ constructed in a part of the problem not reported herein) is consistent with her belief that the value will become zero. At first when she is asked to predict what the graph will look like she hesitates almost to the point of refusing. This is the first time she is asked to predict a graph and she is very reluctant to sketch. She appears to be mentally figuring out values as she slowly marks four points in the first quadrant saying, "I'm not sure. It's going to go down." [6/16]. Keeping the context in mind she then decides how the graph will behave as it crosses the axes and moves into the other quadrants. "I guess it would be, would never cross this line (the y axis). This would be, this right here (the top of the y axis) would be \$4000. That's what you started with. I don't know when it's going to be worthless, but it's not going to cost you money to keep the computer, so it won't go below zero." [6/17]. She connects the points with a sharply decreasing straight line connecting the positive Y axis to the positive X axis. She is not aware that her linear graph is not consistent with the multiplicative changes observed in the values.

She is now asked to make a graph with accurate values and she turns back to the values calculated for the previous parts of the problem and draws a new set of axes. She puts $4000 on the Y axis and 3 on the X axis each located one inch from the origin. She plots the points corresponding to the values of the computer after one, two, and three years, and she calculates and plots the values for years four and five. Because of the small scale the graph is a very gradually decreasing curve. "It looks like it's going to go down gradually, after each decrease." [6/19]. Asked to explain what she means by gradually she says "Well, on the curve this way (to the right) it'll probably level out, but it's not going to be as steep as in the beginning." [6/19].

It is the middle of the sixth interview and they spend the remaining time working on the construction of a compound interest function. Her construction of this function is based on her work with the depreciation problem to the point of mimickry. At the beginning of the seventh session her graph of the depreciation function affects her prediction of the graph of the compound interest function. What Sally remembers is the very gradually decreasing graph, and she sketches a very gradually increasing curve saying, "I know it's going to increase slowly at first, then gradually it's going to get bigger and bigger. [7/2]. ...Each year the percent is even greater, the gain. So that's why it's slow at first and it's going to go up." [7/4]. Now, graphing the compound interest function, she plots enough points to see the exponential shape of the graph which she characterizes as making "a big jump." [7/12]. In the eighth interview she is asked to recall the graph of the depreciation problem and she makes an interesting comparison to the completed graph of the compound interest problem from the seventh interview. Even though she did not take the graph of the computer depreciation problem far enough to see the exponential pattern, she has an intuitive sense of what its graph should be like. "I'm [wondering] if it's going to make the same jump [as the compound interest graph]. I think it's going to level slowly and then really drop all of a sudden. I think it has something to do with the exponents." [8/5]. Interestingly, her prediction here is a reflection of the compound interest graph about a horizontal line rather than an actual depreciation graph. She has a sense of a sudden drop and a leveling off but no sense of when the drop and leveling occur.

Sally's predictions and analyses of these graphs as well as others through the interviews provide evidence that she is looking for ways to describe change and for connections among the various representations of functions. By the end of the nine interviews she is able to differentiate between linear functions and exponential functions in several respects. As our analyses continue we will examine the role of students' ways of representing change in their conceptualizations of exponential functions. More of this analysis will be presented at the conference.

"I HAVEN'T DONE THESE SINCE HIGH SCHOOL": PROSPECTIVE TEACHERS' UNDERSTANDINGS OF MATHEMATICS[1, 2]

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Teacher educators who want to prepare beginning teachers to teach mathematics for understanding may find prospective elementary and secondary teachers' subject matter knowledge inadequate. To support this claim, this paper draws on analyses of interviews with teacher education students at five universities. The discussion focuses on an analysis of teacher education students' knowledge of the relationship between perimeter and area, as well as of their notions about the role of justification in mathematical knowing.

> God, I don't even remember how to do this. (pause) Shoot, no, I have to, I don't know how to do it. Because like I don't know what I'm remembering here that I did . . . . I remember doing these for a long time though and trying to get these down, and so I remember bits and pieces and then I try to apply it generally, and I can't do it. (Elementary teacher education student)

Prospective teachers study little school mathematics content as part of their formal preparation for teaching. When mathematics educators do explicitly teach mathematics content in their methods classes, they often concentrate on nontraditional topics, such as probability or permutations. And, although secondary majors take many college mathematics courses, they rarely revisit the "stuff" of the precollege curriculum, such as ratio and proportion, place value, or perimeter and area. Thus a large part of what prospective teachers will teach is material which they last studied in elementary and high school.

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[2] The author gratefully acknowledges the assistance of other members of the NCRTE research staff, especially Joyce Cain, Ada Beth Cutler, Robert Floden, James Mead, Barbara Neufeld, Pamela Schram, Suzanne Wilson, and Kenneth Zeichner.

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Given this arrangement, can we assume that prospective teachers bring with them adequate understandings of the mathematics they will teach? Researchers at the National Center for Research on Teacher Education have been interviewing prospective elementary and secondary teachers at the point at which they enter formal teacher education (Ball, 1988a, 1988b; Neufeld, 1988; Schram, Wilcox, Lanier, Lappan, & Even, 1988). Results suggest that prospective teachers may not understand even the most fundamental topics--such as place value or division--adequately, nor do they have appropriate understandings of mathematics as a domain of knowledge and inquiry.

While troubling, such results should not be surprising. The algorithmic knowledge fostered in many math classrooms is well-documented (e.g., Davis & Hersh, 1981; Erlwanger, 1975; Goodlad, 1984; Madsen-Nason & Lanier, 1987; Wheeler, 1980). Recent results of the National Assessment of Educational Progress (Dossey, Mullis, Lindquist, & Chambers, 1988) suggest that most pupils do not develop deep and principled understandings of mathematics; the achievement data on seventeen-year-olds is particularly alarming. What we are learning about the subject matter knowledge of prospective teachers simply reinforces what the student achievement data suggests: that relying on what prospective teachers have learned in their precollege mathematics classes is unlikely to be adequate subject matter preparation for teaching mathematics for understanding.[3]

The results below are based on an analysis of the interviews of

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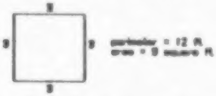
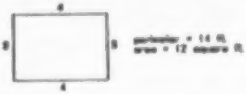
[3] Teaching mathematics for understanding implies that pupils should acquire substantive knowledge of mathematical concepts and procedures along with an explicit emphasis on mathematical ways of knowing. The two are intertwined: In order for students to develop power and control in mathematics, students must learn to validate their own answers. They must have opportunities to make conjectures, justify their claims, and engage in mathematical argument, all of which both depend upon and can extend pupils' understandings of concepts and procedures (Lampert, 1986, 1988).

40 teacher education students--14 secondary math majors and 26 elementary prospective teachers--at five different universities.[4]

One of the questions presented to students[5] during the two-hour interview was the following scenario:

Imagine that one of your pupils comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove that what she is saying is true:





How would you respond to this pupil?

There are two dimensions to this question. The first concerns the specific concepts of perimeter and area and their relationship, the substance of the pupil's claim. The second concerns mathematical knowledge and its justification--"theory" and "proof." The pupil claims to have "discovered a theory" and offers a picture as a "proof." An example, however, does not establish the truth of a generalization in mathematics. The pupil has illustrated, not proven, her claim. Her claim is a conjecture, not a theory.

The data on students' responses to this scenario were analyzed in order to determine how many prospective teachers identified the pupil's "theory" as false or concentrated on the fact that examples aren't proofs, how many thought it was true, and how many were not sure about whether it was true or not. The bases for their responses were analyzed as well. Thus, this analysis focused on two dimensions of subject matter knowledge: (1) truth value and (2) legitimacy (Ball, 1988c). The results are summarized in Table 1; examples of typical student responses follow.

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[4] These data were gathered as part of a longitudinal study of teacher education and learning to teach being conducted at the National Center for Research on Teacher Education at Michigan State University. See NCRTE (in press) for additional information about this study.

[5] I use the term "student" to refer to the college students who are participants in the study and "pupil" to refer to K-12 students.

Table 1
Prospective teachers' responses to a pupil's conjecture about perimeter and area

	Teacher Candidates		
	Elementary	Secondary	TOTALS
Thought the pupil's theory was NOT TRUE	.15	.28	.20
Were NOT SURE whether the pupil's theory was true or not	.50	.28	.43
Thought the pupil's theory was TRUE	.35	.44	.37
Paid explicit attention to PROOF	.00	.16	.10

n = 40

The pupil is right. More than a third of the students (44% of the secondary majors, 35% of the elementary) were impressed with the pupil's accomplishment and accepted the truth of her claim with little question or reflection. One of the math majors, for example, leaned over and, after studying the example briefly, said:

> That's interesting. That would be true, first of all. I'd probably say, "That's a very good observation and that was very neat that you could think of that on your own at home."

Then she added that she would give the pupil another example to illustrate the theory:

> Our classroom, you know, the length of the walls is small, but in the cafeteria, look how long the walls are in the cafeteria. It takes you a longer time to walk around the end of the cafeteria than it does to walk around the walls of the classroom.

Her example illustrates the case in which the perimeter and area are both larger, thus reinforcing, not challenging, the pupil's claim.

The pupil is wrong. Only 20% of the prospective teachers knew that the substance of the pupil's claim was wrong. They said that the asserted relationship between perimeter and area would hold in some cases but that it was not true in other cases. Almost none of these students talked generally about the conditions that would govern the truth of the claim. One of the mathematics majors said he would ask the pupil to search for a case in which the perimeter increases and the area does not; when probed, he could not identify such an example himself. Two other teacher candidates said they would tell the pupil that her "theory" wasn't true all of the time and they would show her a counterexample. One student, searching for such a counterexample, grew less sure that he was right: "I am almost

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I'm not sure if this is right or wrong. Many of the teacher candidates (43%) were unsure whether there was a direct relationship between perimeter and area. Some were not persuaded by the pupil's drawing but could not remember if the assertion was true either. These teacher candidates said they would have to figure it out for themselves. One of the math majors, for example, said she would test it for herself, "to see if it's true--if I came up with an example where it wasn't true, then I would point that out to her."

Others who weren't sure said they would look it up or ask someone. They seemed less inclined to puzzle over it themselves. Their uncertainties sometimes stemmed from not remembering the relevant concepts as well as not knowing whether the relationship held. For example, one student wasn't sure at first what *area* was: "What *is* the 'area'? Is it the *inside*?" Then she said that it seemed like "a reasonable assumption if you're going to increase the parameter (sic), of course the area inside is going to increase."

Examples, proof, and the justification of knowledge. Perhaps most significant was that, in responding to this question, 92% of the elementary and 86% of the secondary prospective teachers focused exclusively on the *substance* of the student's claim and responded in terms of what *they* knew about perimeter and area and the relationship between them. They did not comment on the pupil's approach to establishing her claim; instead, they were preoccupied with whether she was right or wrong.

Not only didn't the teacher candidates worry about the insufficiency of the "proof," but a few prospective teachers even seemed to become convinced themselves by the pupil's drawing. They looked closely at the sketch; a few carefully checked the calculations of perimeter and area. They made comments like "it looks like it" or "it seems right."

Several teacher candidates were pleased that the pupil had figured something out for herself. One said it was good that the pupil had "put the two pieces together, especially on her own," that many pupils "never really tie it together"--they just "get this piece and that piece." Another said she would tell the pupil, "You are really using your thinking skills about mathematics." These teacher candidates were preoccupied with the pupil doing something "on her own," a preoccupation common in mainstream American culture. An emphasis on ownership, individuality, and independence can jeopardize, as it does in this case, concerns for legitimacy. "Thinking on your own," if it is wrong, is in direct tension with acquiring disciplinary knowledge and skills. Yet most of the teacher candidates seemed to be oblivious to the problem of justifying knowledge claims on the basis of one example. To say that to prove something by showing an illustration is to use "thinking skills in mathematics" completely misses the point of what mathematical thinking entails.

CONCLUSION

To assume that the content of first or fifth or ninth grade mathematics is something any high school graduate understands is to doom school mathematics to a continuation of the dull, rule-based curriculum that is so widely criticized. Throughout our interviews, many college students, including those who were majoring in mathematics, had difficulty working below the surface of so-called "simple" mathematics (e.g., place value, division, simple equations). Although many (not all) could perform the procedures, they seemed to lack legitimate understanding of the content: If they could remember something, they often could not explain or justify its basis--convention and logic were frequently blurred (Ball, 1988c; Lampert, 1988). Few seemed to have developed an appropriate sense of what doing mathematics entails, such as the role of conjecture and argument.

Close analyses of the mathematics entailed in division of fractions, of zero, and in algebra (Ball, 1988c), as well as of what is involved in mathematical discourse (Lampert, 1988) show that elementary content, if taken seriously, is anything but simple. Duckworth (1987) refers to the "depths and perplexities of elementary arithmetic" and in her writing, as well as in Lampert's (1985, 1986, in press), the content of the school curriculum is opened up and its mathematical complexity revealed. Teacher educators may be able to convince prospective teachers that "teaching for conceptual understanding" should be the goal. However, without revisiting the "simple" mathematical content they will teach--to revise and develop correct understandings of the underlying principles and warrants, of the connections among ideas--prospective teachers may be wholly unprepared to do more than teach "invert and multiply" "foil," or "cross-multiply and divide."

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EVALUATION OF THE MATHEMATICAL BACKGROUND OF PROSPECTIVE ELEMENTARY SCHOOL TEACHERS[1]

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The evaluation of the mathematical background of 209 prospective primary school teachers was assessed through an examination which consisted of two parts, one involving the solution of problems, the other involving knowledge relevant to the solution of the given problems. We have found evidence that on geometry and proportional reasoning a score of 75% on the relevant knowledge part seems to be a threshold for a significant rate of success on the solution of the given problems. But even for those in the 75-100% range on relevant knowledge, the failure rate on the solution of the given problems is 34.6% and 74.3% respectively. Relating success on problem solving with prior course work, shows that even with 1 or 2 collegial mathematics courses, the failure rate remains high at close to 50%.

Although the primary school teacher spends from 25 to 30% of her classroom time on the teaching of mathematics, out of the 90 credits obtained in her three years at the Université de Montréal, barely six credits (7%) on the psycho-pedagogical aspects of mathematics instruction are compulsory. Students entering the Bachelor's program in Elementary School Education constitute a rather heterogeneous group with respect to their mathematical training, some having taken up to five courses at the level of collegial mathematics, while others have remained at the Secondary IV level[2]. This situation is similar to that found in other parts of Canada and the United States. For mathematics educators involved in teacher training, having so little time to prepare their prospective teachers means that it is of prime importance to determine, right from the start, whether or not they have acquired the necessary mathematical background to teach more meaningfully the basic mathematical concepts taught at the primary school. To find an answer to this question, we have constructed an entrance examination assessing their mathematical knowledge and their utilization of this knowledge.

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This paper has two main objectives. The first one is to describe the innovative nature of the exam we have designed and to explain the underlying rationale. For instance, we have distinguished between problem solving and the pre-requisite knowledge needed to solve each problem; we have also established criteria to evaluate the mathematical reasoning used in problem solving, regardless of whether the answer found was right or wrong. Our second objective is to report the results obtained by the 231 first-year students who sat for the examination in the Fall of 1987.

DESIGN OF THE EXAM

Since we only had 90 minutes available, we had to limit ourselves to five topics: geometry, proportional reasoning, fractions, simple algebra, and familiarity with the pocket calculator. For each one of these, we were not interested in assessing scattered pieces of knowledge, but rather in judging how relevant knowledge would be used to solve problems. For each of the topics, we designed a typical problem and then determined what mathematical knowledge was essential for its solution.

The two parts of the exam were each worth 50% of the total mark. For the part on relevant knowledge, in order to avoid any guessing, we opted for a non-multiple-choice format that required the generation of answers. For the problem solving part, the prospective teachers were told that 75% of the marks was allocated to reasoning and 25% for the right answer. We now report the results based on 209 of the 231 subjects for whom sufficient data is available.

RESULTS ON SOME TOPICS

Since space is limited, we will present full details on the two topics, geometry and proportional reasoning, which have provided us with the most interesting results.

Geometry problem. A square, an equilateral triangle (three equal sides), and a circle, are each enclosed by a 24 m fence. Do you think that they all have the same area? If not, which one has the largest? the smallest? Justify your answer.

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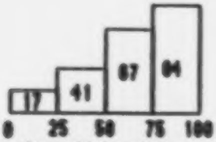
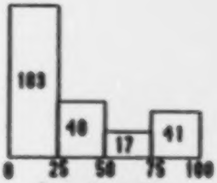
Relevant knowledge. 1. What is the formula for: the perimeter of a square, rectangle, triangle; the circumference of a circle; the area of a square, rectangle, triangle and circle. 2. If the perimeter of a square is 60 cm, what is the length of a side? 3. What is the length of the hypothenuse (the longest side) of the following right angle triangle (base 6, height 3)? 4. Find the values of π and √2 (to three decimal places).

The following histograms show the great contrast existing between the distribution of the marks obtained in the solution of the geometry problem and the distribution obtained on geometric pre-requisite knowledge. Each cell indicates the number of prospective teachers scoring within the given range.



Geometric knowledge



Geometry problem

It is remarkable that 72.3% ((67+84)÷ 209) of the prospective teachers achieve more than 50% on the knowledge test but that only 27.8% ((17+41) ÷ 209) succeed in the solution of the problem. This is despite the fact that partial marks were given for each step in the solution of the problem. The gap between relevant knowledge and problem solving becomes even more visible in the following table which shows the problem solving distribution (P.S.) for the four subgroups defined according to their knowledge. The numbers in parentheses represent the percentage of the number in the cell relative to the total in the column which can be seen in the preceding histogram on the left.

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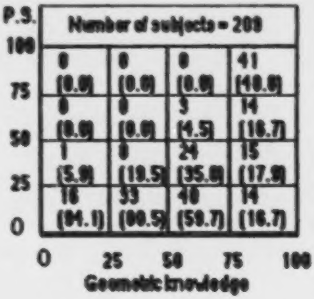
The results shown in the following table lead to some startling observations. The first one is that hardly anyone (3 subjects) with less than 75% on the geometric knowledge test manages to achieve 50% on the solution of the problem. It would seem that 75% is the threshold at which significant problem solving can succeed.


P.S.
100
75
50
25
0
0
25
50
75
100
Geometric knowledge

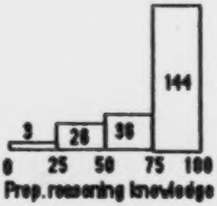
Number of subjects = 209			
0 (0.0)	0 (0.0)	0 (0.0)	41 (48.8)
0 (0.0)	0 (0.0)	3 (4.5)	14 (16.7)
1 (5.9)	8 (19.5)	24 (35.8)	15 (17.9)
16 (94.1)	33 (80.5)	40 (59.7)	14 (16.7)

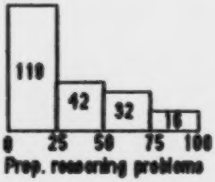
The second observation is that even among the 84 prospective teachers who score between 75 and 100% on the knowledge part, 29 of them (34.6%) fail to obtain 50% on the solution of the problem.

Problems on proportional reasoning. 1. If a merchant charging $4.00 for a pound of tea looses 20% on his cost, how much should he sell it in order to make a profit of 30% on his cost? 2. If 12 people can accomplish a given job in 60 days, how long will it take for 17 equally experienced people to complete the same work?

Relevant knowledge. 1. Convert into common fractions: 25%, 125%, 0.125, 1.4. 2. What percentage of the chips are black (3 black, 4 white)? 3. What number needs to be inserted in the equation 3 / 8 = □/24 ? 4. Evaluate 37% of 875; 5 / 7 of 613.24.

The two histograms on proportional reasoning are even more symmetric than the ones on geometry. Here, the two success rates are even more extreme:





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86.1% ((36+144) + 209) achieve 50% or more on the knowledge part and 23.0% ((32+16) + 209) on the solution of the problems. The next table shows the problem solving distribution (P.S.) for the four subgroups according to their knowledge.

Number of subjects = 209

P.S.	0-25	25-50	50-75	75-100
75-100	0 (0.0)	0 (0.0)	3 (8.3)	13 (9.0)
50-75	1 (33.3)	3 (11.5)	4 (11.1)	24 (16.7)
25-50	0 (0.0)	5 (19.2)	4 (11.1)	33 (22.9)
0-25	2 (66.7)	18 (69.2)	25 (69.4)	74 (51.4)

Prop. reasoning knowledge

That the acquisition of relevant knowledge is not sufficient to solve the problems is confirmed here too: although 144 prospective teachers are in the 75-100 quarter on the knowledge part, 107 of them (74.3%) fail on the problem solving part. One of the causes of this very high failure rate can be attributed to a blind application of proportional reasoning to the second problem and a failure to perceive that inverse proportions are involved. The above table also shows some unexpected data. Of the 29 subjects who fail the knowledge part, 4 of them succeed on the problem solving part. This seemingly contradictory result could perhaps be attributed to the easy numbers chosen for the first problem. The price of a pound of tea at $4.00 resulting in a 20% loss might have induced some subjects to guess the actual cost without having to rely on the use of proportions. The 36 subjects in the third column also provide some surprises: three of them score higher on problem solving than on relevant knowledge. A likely explanation is that the two problems could be solved without relying on all the knowledge tested. For instance, the conversion of a percentage into a common fraction can be viewed as relevant to proportional reasoning, but is not needed to solve the problems.

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Overall examination results

The data shown in the table to the right lead to some very clearcut conclusions. The emptiness of the six cells in the upper left corner above the diagonal corresponds to our expectation that in general, problems can only be solved when the required knowledge is present. A trend we saw in the geometry results seems to be confirmed here at a more general level. The table indicates again that a 75% level in overall knowledge seems to be a threshold for a significant rate of success in problem solving. Among the 59 subjects who obtain between 50 and 75% on the knowledge part, only 11 of them (18.6%) succeed in solving problems, and at that quite moderately, while a failure rate of 81.4% prevails for this subgroup. On the other hand, these rates are reversed when we examine the subgroup scoring between 75 and 100% on overall knowledge. Of the 120 subjects, 23 of them (19.2%) fail on problem solving, while 80.8% of them succeed.

Number of subjects = 209

P.S.	0	25	50	75
100	0 (0.0)	0 (0.0)	0 (0.0)	33 (27.5)
75	0 (0.0)	1 (4.0)	11 (18.6)	64 (53.3)
50	0 (0.0)	6 (25.0)	36 (61.0)	22 (18.3)
25	5 (100)	18 (72.0)	12 (20.4)	1 (0.8)

0 25 50 75 100
Overall Results

Impact of prior courses

The following tables provide the distribution of scores relative to the number of prior mathematics courses. Ten of the 209 subjects did not supply information regarding their background, which leaves us with 199 subjects. The last row indicates the failure rate (F.R.).

The table on relevant knowledge provides us with a failure rate of 34.2% for those subjects who stopped at high school courses, (Sec.IV and V) compared with 8.5% for those who took mathematics courses at College. The ratio between these two rates is about the same as that for the total examination scores. The failure rates for those

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having taken one or two collegial courses is nevertheless surprisingly high (18.2% and 11.4% respectively). But these failure rates pale when compared to those on the solution of problems. For the same two groups, the rates are 75.7% and 31.0% respectively. The last average rate hides the much higher rates obtained for those with one or two collegial courses (48.5% and 48.6% respectively).

Scores on relevant knowledge

	Sec IV	Sec V	1	2	3	4	5	
N=	10	60	33	35	28	27	6	=199
100	3	18	17	25	27	26	4	
75	2	23	10	6	1	0	2	
50	5	16	5	4	0	1	0	
25 0	0	3	1	0	0	0	0	
F.R.	50.0	31.7	18.2	11.4	3.7	0.0	-	

Number of math courses at College

Scores on the solution of problems

	Sec IV	Sec V	1	2	3	4	5	
N=	10	60	33	35	28	27	6	=199
100	1	2	3	3	8	14	1	
75	1	13	14	15	15	12	4	
50	1	25	14	15	5	0	1	
25 0	7	20	2	2	0	1	0	
F.R.	80.0	75.0	48.5	48.6	17.9	3.7	-	

Number of math courses at College

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CONCLUSIONS

The design of the entrance exam reflects the importance that is now universally attached to solving mathematics problems. Attributing marks for evidence of reasoning provided us with a more refined judgment of the subjects' performance. The weight given to reasoning compared to that given for the answer (75% vs 25%) also conveyed our bias for mathematical thinking. We do seem to have gathered evidence indicating that around a 75% level of knowledge relevant to a problem needs to be attained before we witness significant success in the solution of that problem. This was shown for the problems in geometry and in proportional reasoning. It was also confirmed more globally in the analysis of the overall examination results. Of course, the 75% threshold on knowledge does not insure success in the solution of the problems since at the global level, about 19% of those having exceeded it still failed on problem solving. Another interesting result stems from the relationship between success rate on the exam and prior coursework. It is most surprising to find that even with two collegial mathematics courses, the failure rate on the solution of simple high school problems remains close to 50%. Whatever is learned in these courses certainly does not seem to help them solve these problems. The evaluation of the mathematical background of prospective elementary school teachers is definitely an important issue. The approach we have used in designing our exam has yielded promising results. However, this area of research could greatly benefit from a collaborative effort by educators from different universities. This might result in a better evaluation instrument and more generalizable results.

Notes:

1. Research funded by the Quebec Ministry of Education (F.C.A.R. 89-EQ2923) and the Faculty of Education of the Université de Montréal.
2. The Quebec educational system involves 6 years at the primary level, 5 years at the secondary level, 2 in junior college and 3 at the university for a Bachelor's degree.

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PRE-SERVICE TEACHERS CONCEPTIONS OF THE RELATIONSHIP BETWEEN FUNCTIONS AND EQUATIONS

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The study reported here is part of a larger study on prospective secondary math teachers' knowledge and understanding of functions and teaching functions. The report here concentrates on their knowledge and understanding of the relationships between functions and equations. The teachers were asked to write a definition of a function, to indicate how functions and equations are related to each other, and to find the number of solutions to a quadratic equation, given a positive value and a negative value of the quadratic expression. The findings suggest that many of these students hold a limited view of functions as equations only, and lack the ability to relate solutions of equations (algebraic representation) to values of a corresponding function (in graphic representation).

INTRODUCTION

Function is a central concept in mathematics. Studies of students' understanding of function point to difficulties in the learning of this concept and show that many students hold a very limited concept. Three major components directly influence the learning process : the learner, the curriculum and the teacher. Teachers' subject-matter knowledge influence teaching and therefore students' learning. The purpose of this study is to explore aspects of teacher subject-matter knowledge of the concept of function. Main areas of teacher subject-matter knowledge are subject-matter content knowledge, a common personal knowledge of the subject-matter, and pedagogical content knowledge, knowledge of how to help someone else develop an understanding of the subject-matter (e.g., Shulman, 1986).

This study concentrates on the following aspects of each one.

Subject matter content knowledge:

1. What is a function? (includes image and definition of the concept of function, univalent property of functions, and arbitrariness of functions).

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2. Inverse function and composition of functions.

3. Different representations.

4. Functions from the high-school curriculum.

5. Different ways of approaching functions: point-wise, interval-wise, global and as entities.

6. Procedural and conceptual knowledge.

Pedagogical content knowledge:

1. Students' mistakes -- what they do and why.

2. Procedural and/or conceptual teaching.

BACKGROUND OF THE STUDY

We will deal here with two aspects of the relationship between functions and equations (related to 1, 3 and 4 above from subject matter content knowledge). One has to do with the role of equations in the definition and image of the concept of function. The other has to do with the relationship between the values of a function and the solutions to an equation.

a) The role of equations in the definition and image of functions.

The definition of functions has changed during the last two centuries. While an 18th century function was an analytic expression, representing the relation between variables, with its graph having no "sharp corners" (referred to as Euler's definition), in the 20th century function came to be a subset of the Cartesian product of two sets in which each member of the domain is paired with exactly one element of the range , or an alternative and less formal definition: A correspondence between two sets that assigns to each element from the first set exactly one element from the second set (Dirichlet-Bourbaki). Two hundred years ago functions were equations that described the relation between two variables using expressions. Today's definition of functions is not so limited. Functions do not have to be graphable, to be representable by equations, and their domain and range may be sets of objects other than numbers. Malik (1980) claims that Euler's definition from the 18th century covers all the functions used or required in a calculus course, and up to this stage one never confronts a situation where one has to use the modern definition of function. Malik concludes that since only a particular form of functions is used, the student unconsciously accepts this particular form as the definition. Other studies also show that while students are being taught the modern definition of function, the old one serves as the concept image for these students (Marnyanskii, 1965; Markovitz et al, 1986; Vinner and Dreyfus, in press). The modern definition is used in current texts, so secondary teachers are expected to teach functions that way. What is the concept image of functions for them? Is it the earlier one or is it the modern one? What will they teach their students; and why?

b) Values of functions as solutions to equations.

Conceptual knowledge is knowledge that is rich in relationships (Hiebert and Lefevre, 1986). Sometimes it is easier to solve an equation by looking at the corresponding function (if it exists) and relating the solutions to the graphical representation. For example, the solutions of $a_nx^n + \ldots + a_1x + a_0 = 0$ are the x-intercepts of the graph of the function $f(x) = a_nx^n + \ldots a_1x + a_0$. Flexibility in moving from one representation to another allows one to see rich relationships and to develop a better conceptual understanding. We especialy would want to see this flexible translation from one representation to another in the case of dealing with basic common functions from the high school curiculum such as the quadratic function. So, when given a familiar algebraic expression in a problem situation it is desirable for students (and teachers) to be able to make connections between the expression and its corresponding function's graphical representation. How flexible is the teachers' understanding of function? How rich in connections between different representations is their knowledge?

METHOD

The subjects of this study were prospective secondary mathematics teachers in the last phase of their professional education. This group was selected so that the description of their knowledge would reflect the knowledge teachers have gained during their college education, but before they start real teaching in the field. The subjects came from eight mid-western universities in the U.S.A. who agreed to participate in the study.

Data was gathered in two phases. At the first phase, an open-ended questionnaire was administratered to 152 subjects. Information gathered from a written questionnaire is sufficient for a general description of some facets of the teachers' knowledge, but is limited and sometimes hard to interpret. In order to overcome these difficulties the second phase included interviews with additional ten subjects. By probing, asking subjects to explain what they did and why, asking for their reactions as future teachers to students' misconceptions and asking questions which are related to the questionnaire but require more general, longer or more thoughtful responses, a more accurate and detailed picture of the subjects' subject-matter content knowledge and pedagogical content knowledge was developed.

RESULTS

The results presented are based primarily on the questionnaire data.

a) The role of equations in the definition and image of functions.

Several questions were designed to investigate what function is for the prospective teachers. The first one was: *Give a definition of a function* . Some of the answers were correspondence, mapping, relation, rule, operator and graph. Thirty three subjects (out of 146 who answered this question) defined function as an equation, an algebraic expression or a formula. A strict categorization was employed to subjects' responses. Unless the words "equation", "algebraic (numerical) expression", or "formula" were mentioned, responses were not included.

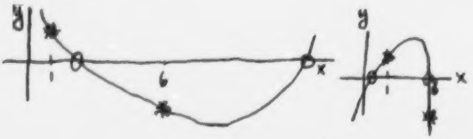
The question: *How are functions and equations related to each other?* , caused more subjects to identify functions with equations. An additional 26 subjects said that functions are equations or that rules for functions are equations (without any additional remarks that some functions may not be representable by equations).

Together, almost half (59 out of 126) of the subjects who answered both questions related functions directly to equations in at least one of their answers. This tendency to look at function as an equation only was also shown during the interviews. Although the prospective teachers were aware of other representations of function such as graph and arrow diagram, these representations seem to have weak interconnections in the subjects mind. This was true even for very familiar expressions such as the quadratic expression as we can learn from the next section.

b) Values of functions as solutions to equations.

When asked the following question: *If you substitute 1 for x in* $ax^2 + bx + c$ *(a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation* $ax^2 + bx + c = 0$ *have? Explain.* Fourty five subjects (out of 127 who answered this question) gave the correct numerical answer: 2. But only 18 subjects (14%) got this answer by referring to the graph of a quadratic function and using the Intermediate Value Theorem:

"2. This is a parabola. The graph will be either ∪ or ∩ depending on if a is + or –. If x = 1, then y is above the x-axis. If x = 6, y is below the x-axis. It therefore must cross the x-axis. The parabola is symmetric, so there will be 2 x-axis intercepts."



Eight subjects used the intermediate value theorem but did not consider the special given function and therefore found only one real solution.

A large group of subjects (19) claimed that the number of real solutions is two since a polynomial of degree 2 has two real solutions!

— "2 real solutions because it has an order of 2".

About the same number of subjects (20) just "played" with inequalities without reaching any conclusion.

— "$a + b + c > 0$ $36a + 6b + c < 0$ $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

the solution is real when $\sqrt{b^2-4ac}$ is real"

— "$a + b + c > 0$ $36a + 6b + c < 0$ $a + b + c > 36a + 6b + c$

$0 > 35b + 5a$

$-35a > 5b$

$-7a > b$

$a < -1/7b$"

Nine subjects considered a, b, c to be variables and reached the conclusion that there are an infinite number of solutions:

— "$(a + b + c) > 0$ $(36a + 6b + c) < 0$ $a < 0$.

(∞) many - dependent on changes in b and c".

DISCUSSION

A strong tendency to identify functions with equations or expressions was detected among the prospective teachers. Even if, as some might claim, knowing the distinction between these is not important for the functions taught in secondary school, the teacher is expected to teach the modern definition of function. The teachers incomplete conception of function is problematic and may contribute to the cycle of discrepancy between concept definition and concept image of functions in students.

A lack of rich relationships, which characterize conceptual knowledge, seems to prevent the prospective teachers from relating the given equation: $ax^2 + bx + c = 0$, to a graphical representation of the function $f(x) = ax^2 + bx + c$. Eighty percent of the subjects did not make the connection. Eisenberg and Dreyfus (1986) report similar findings when in a course which stressed the graphical method of solving inequalities, only 5% of the college students opted for the graphical solution on the exam. Another interesting finding shows that more than 1/4 of the prospective teachers gave answers which don't make sense at all, such as ∞ and even 3, 4 or 5 as the number of solutions of a quadratic equation. Although they probably "knew" that a quadratic equation has at most two solutions, they were not able to use it in a situation that requested dealing with seemingly conflicting information.

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GEOMETRIC UNDERSTANDING AND MISCONCEPTIONS AMONG PRESERVICE AND INSERVICE MATHEMATICS TEACHERS

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This study examines the level of geometric understanding and misconceptions among preservice elementary teachers, preservice mathematics teachers, and inservice secondary mathematics teachers. Most preservice secondary mathematics teachers were found to be operating at the van Hiele deduction level or above, but many exhibited misconceptions, especially in the concepts of parallelism. Inservice secondary mathematics teachers exhibited few of these misconceptions while generally operating at the highest van Hiele level. In contrast, the van Hiele levels of elementary preservice teachers were quite low, with approximately 20% not adhering to the hierarchical pattern of the van Hiele levels.

Recent curricular reforms mandated by several national and state organizations have placed increased emphasis on geometry in the mathematics curriculum at all levels. This study examines the level of geometric understanding and misconceptions among preservice elementary teachers, preservice secondary mathematics teachers, and inservice secondary mathematics teachers.

The van Hiele Model of Geometric Understanding

Dutch educators Pm M. van Hiele and Dina van Hiele-Geldof proposed a linearly-ordered model of geometric understanding. The van Hiele theory asserts that there exist five hierarchical levels of geometric thinking that a successful learner passes through: visualization, analysis, abstraction, deduction, and rigor. These levels as described in behavioral terms are:

> **Basic Level:** At this level, figures are recognized by appearance alone. A figure is perceived as a whole, recognizable by its visible form, but properties of a figure are

> not perceived. At this level, a student should recognize and name figures and distinguish a given figure from others that look somewhat the same.
>
> **Level I:** Here, properties are perceived, but they are isolated and unrelated. Since each property is seen separately, no relationship between properties is noticed and relationships between different figures are not perceived. A student at this level should recognize and name properties of geometric figures.
>
> **Level II:** At this level, definitions are meaningful, with relationships being perceived between properties and between figures. Logical implications and class inclusions are understood. The role and significance of deduction, however, is not understood.
>
> **Level III:** At this level deduction is meaningful. The student can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. A student at this level should be able to supply reasons for steps in a proof.
>
> **Level IV:** The student at this level understand the formal aspects of deduction. Symbols without referents can be manipulated according to the laws of formal logic. A student at this level should understand the role and necessity of indirect proof and prove by contrapositive (Mayberry, 1983, p. 59).

According to the van Hieles' model, the learner cannot achieve one level without passing through the previous levels. Progress from one level to the next is more dependent on educational experiences than on age or maturation, and certain types of experiences can facilitate (or impede) progress within a level and to a higher level.

Related Research

Joanne Mayberry (1981) found sufficient evidence among 19 undergraduate preservice elementary teachers to support the hierarchical nature of the van Hiele levels. She rejected the hypothesis that an individual demonstrated the same level of thinking in all areas of geometry included in the school program. The van Hiele levels of her subjects were quite low: they did not recognize squares as rectangles and did not perceive relationships between classes of figures. Those subjects who had taken high school geometry did seem to achieve at a higher level than those who had not.

Two recent studies indicate that teachers and students tend to exhibit the similar patterns of misconceptions. Fuys, Geddes, and Tischler (1985) interviewing several populations reported misconceptions or errors found among 6th and 9th graders appeared also with elementary inservice teachers. Among the examples cited were thinking "sides" refers only to vertical segments, using the phrase "straight lines" when referring to parallel lines, and thinking that a parallelogram has to have obliques angles (Fuys, Geddes, and Tischler, 1985, p. 199). Hershkowitz (1987) found that the geometric misconceptions displayed by 142 preservice elementary teachers and 25 inservice elementary teachers were repeated in students in grades 5 through 8. Examples include misidentification of right triangles, isosceles triangles, quadrilaterals and altitudes in various types of triangles.

The Present Study

The present study focuses on the level of geometric understanding and misconceptions among different teacher populations: preservice elementary teachers, preservice secondary mathematics teachers, and inservice secondary mathematics teachers, and inservice secondary mathematics teachers. The population consists of 67 preservice undergraduate elementary teachers enrolled in an informal geometry course for elementary education majors with mathematics as an area of concentration; 10 preservice undergraduate middle school teachers enrolled in a middle school mathematics methods course; 26 preservice undergraduate elementary school teachers enrolled in an elementary mathematics methods course; 18 preservice secondary mathematics teachers enrolled in a secondary mathematics methods course which included topics from both Euclidian and non-Euclidian geometry; and 27 mathematics teachers enrolled in graduate level mathematics education courses.

The van Hiele level of all subjects was determined from a 25 item paper-and-pencil test developed by the Cognitive Development and Achievement in Secondary School Geometry Project (Usiskin, 1982). In addition, 38 of the preservice teachers participated in a one to two hour interview based on Mayberry's questions. Selected questions of particular interest from the interview were administered in written form to the inservice teachers. The paper-and-pencil and interview questions focused on the concepts of square, isosceles triangle, right triangle, circle, parallel lines, similarity, and congruence.

Analysis of the protocols indicates several patterns. Over three-fourths of the secondary mathematics preservice teachers are operating at the deduction level (Level 3) or above. Although they can construct proofs, understand the role of axioms and definitions, know the meaning of necessary and sufficient conditions, and manipulate symbols according to the rules of formal logic, many of the subjects exhibit misconceptions. These misconceptions may be due, in part, to incomplete or faulty recall of definitions and theorems. Often the precise conditions for a definition of a theorem were ignored or forgotten. For example, several of the secondary preservice teachers stated that parallel lines do not have to lie in the same plane in Euclidian geometry. Several defined isosceles triangles as having exactly two sides equal, but applied well-known isosceles triangles theorems such as the angles opposite two congruent sides of a triangles are congruent to equilateral triangles without hesitation. The experienced inservice secondary mathematics teachers generally operated at the highest van Hiele level. They exhibited few of these problems, perhaps because they teach such definitions and therefore the definitions are both fresher and more well defined in their minds.

Both the preservice and inservice secondary teachers constructed congruence proofs with ease and most exhibited an understanding of indirect proof. The preservice secondary teachers had difficulty determining what was being proven if they were shown a proof without a statement of the theorem being proven. For example, when shown the following question: *Line DE is parallel to line AB in triangle ABC intersecting the triangle at vertex C> Because of the properties of parallel lines we can prove that <DCA is congruent to <CAB and <ECB is congruent to <CBA. Now, <DCE is a straight angles (180 degrees). What have we proved?* most preservice secondary mathematics teachers either didn't know or thought that they had proved that

the alternate interior angles formed by a transversal cutting two parallel lines are congruent.

In contrast to the high van Hiele levels exhibited by the secondary preservice and inservice mathematics teachers, the van Hiele levels of the elementary preservice teachers were quite low with 38% functioning below the deduction level and , in fact, 8% not even mastering the lowest level, visualization.

Relationships were particularly troublesome. Several elementary preservice teachers identify rectangles as squares. The preservice groups contained many subjects who identified isosceles triangles as triangles having exactly two sides equal, stated that an isosceles right triangles was an impossibility, and stated that rectangles must be wider than they are tall. Most elementary preservice teachers could not differentiate between necessary and sufficient conditions. Similarity also was problematic, with many subject failing to maintain the invariance of the corresponding angles.

However, several subject from the preservice groups were able to answer questions typically requiring understanding and though at the deduction level while unable to answer questions at a lower level. Approximately 28% of both the preservice secondary and preservice elementary teachers did not fit into the hierarchical pattern of the van Hiele levels. If the Level 4 is disregarded (p. M. van Hiele himself has questions the existence of this as a separate level.), only 6% of the secondary preservice teachers fail to fit the hierarchical pattern. In this case 20% of the elementary preservice teachers still do not conform to the hierarchy.

Summary

In summary, the inservice secondary mathematics teachers tested functioned at the top two level of the van Hiele hierarchy. They can construct proofs, make generalizations, understand the role of axioms and definitions, know the meaning of necessary and sufficient conditions, and manipulate symbols according to the rules of formal logic. While the secondary preservice mathematics teachers tested functions at the van Hiele levels almost comparable to those of experienced teachers, they displayed several misconceptions due, in part, to incomplete or faulty recall of definitions and theorems. The experienced teachers tested had mastered these definitions and theorems, perhaps due to their involvement in teaching them.

The findings concerning the elementary preservice teachers' understanding of geometric concepts are consistent with those previous reported by Mayberry (1981, 1983); Fuys, Geddes, and Tischler (1985) and Hershkowitz (1987). A higher proportion of these preservice teachers do not conform to the van Hiele hierarchy of levels in this study. These data are inconsistent with the van Hiele hypothesis that they learning cannot achieve one level without passing through the previous levels. However, since this study deals with the adult learner, it is possible that these subjects have achieved the types of thought characteristic of the lower levels of the model and that these achievements are not evident due to misconceptions stemming from faulty recall of precise definitions, etc. Perhaps they are capable of visualization, analysis, and abstraction, but are being misled by their faulty prototypes.

Implications for Teaching and Research

The secondary inservice mathematics teachers tested were functioning at a high level of geometric understanding and appeared to possess the background to be excellent geometry teachers. The secondary preservice mathematics teachers, while not functioning at quite the proficiency level of their experienced counterparts, appear to have the geometric understanding to adequately teach geometry. The refresher afforded by preparing lessons may be sufficient to provide the necessary review of concepts and definitions.

The preservice elementary teachers are not as well prepared, with 38% functioning below the deduction level and 8% not even mastering the lowest level, visualization. Since Fuys, Geddes, and Tischler (1985) and Hershkowitz (1987) indicate that in grades 6 - 9 students and their teachers tend to exhibit similar patterns of misconceptions, perhaps we have reason to be concerned. Including more geometry in the preservice elementary program may be in order. In addition, further research at lower grade levels is indicated.

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A CASE STUDY OF A SIXTH GRADE TEACHER'S IMPLEMENTATION AND EVALUATION OF LESSONS FOR CHILDREN IN THE LEARNING OF DECIMALS

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This case study, an extension of an earlier one, reports the monitoring of the implementation of a six-week unit on decimals developed and taught by Janet, a sixth-grade teacher, participating in the Rutgers-Kenilworth Teacher Development Project.[1] The objective was to document the extent to which her lessons reflected the goals of the project. Observations of the lessons indicated: (1) evidence of Janet's creation of mathematical environments in which children, working alone or in groups, were engaged in activities designed to connect symbolic and concrete modes of representation, and (2) that Janet incorporated into her instruction the input she was receiving from children's interaction with her and each other.

The Rutgers-Kenilworth Teacher Development Project in Mathematics was intended to assist the classroom teachers to develop the knowledge, confidence and skills for creating classroom environments in which children could construct their own understanding of mathematics. One aspect of assessing understanding, from this perspective, is for teachers to consider the child's construction of a representation of the structure of the problem and connect these with other modes of representation (e.g., concrete, pictorial and symbolic). In this setting, the teacher's role is often as a guide or facilitator of learning. She provides explorations and activities with appropriate materials that encourage children to build multiple representations. The goal of this approach is to motivate the mathematics activities through real-life, problematic situations.

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[1] The Rutgers-Kenilworth Project is sponsored by the Rutgers University Center for Mathematics, Science, and Computer Education, the Kenilworth School District and the New Jersey Department of Higher Education.

An earlier study (Maher and Alston, 1988) examined a series of four lessons given by Janet over a six-month period which documented the chronology of her progression from a direct instruction teacher to one who designed lessons that encouraged individual and group problem solving. It illustrated Janet moving from the first lesson, in which she found herself showing the children how to construct representations for a particular problem, to the final lesson in which the children worked freely in groups solving non-traditional, multi-step problems with very little teacher input.

During this six-month period, Janet began to test the use of small group activities in her class. In observing a videotape of a small group problem-solving session in which children were constructing, on a geoboard, figures of particular shapes covering a given percentage the board, she noted that children worked together, shared their strategies, and questioned their own as well as other group members' solutions. She also observed children's increased willingness to explain answers, in her absence, to doubting members of the group. While viewing a videotape of a group of her students working on a math activity, she expressed surprise that even the children ordinarily requiring remedial help had found ways to construct solutions to the problems. In her evaluation she stated that in the past, "children learned the formula, $V = L \times W \times H$. Children were never given a chance to construct the figures." In the end, she stated that the children seemed to have a new found confidence in solving these types of nontraditional (multi-step) problems independent of her guidance and assistance.

This paper continues the documentation of Janet's growth with this approach to teaching.

METHODOLOGY AND DATA SOURCE

Two videotapes of each classroom session were made during the period of the study. One camera traced the activities and interaction of a group of children and the other followed the teacher and the class. At the conclusion of the 26 days of lessons, Janet was asked to select a set of lessons which in her judgement reflected her goals for teaching and learning mathematics. This paper is based on videotaped excerpts from three of those lessons, each of which was analyzed by two graduate researchers to identify segments that documented certain teacher and student behavior using an instrument designed for that purpose. The transcribed segments, observers' notes, and children's written work provide the data for this paper. Space constraints do not permit the presentation of all the excerpts and their analysis. However, a detailed account is available (Purdy, 1987).

RESULTS

The purpose of the lessons was to encourage the development of the student's understanding of decimals by providing them with activities that enabled them to make the connection between concrete and symbolic representations for decimals. Decimal squares and decimal playing cards were used extensively as the concrete embodiment for decimals and were used within a variety of settings including group activities, individual seat work, and entire class discussions.

To evaluate the lessons, Janet moved from group to group, asking questions, and providing additional exercises for her students. By examining both their written work and responses to the questions she asked, she attempted to determine their level of understanding and make decisions regarding the direction of each lesson based on this monitoring of student progress. The following excerpts from selected lessons provide a few examples of these activities and the environments created by Janet and the resulting class interactions and discussions.

Lesson One. The purpose of this lesson was to give children the opportunity to make a connection between the symbol for a decimal number and some representation of it. Exercises were provided for children working in small groups using decimal squares and playing cards to represent decimal numbers in various combinations. The following is an excerpt describing the interaction between students and students, and students and teachers that occurred while the children worked. Janet observed one group having trouble determining what to use to represent a "whole," so she decided to open the discussion to the entire class.

Janet: Are any of the playing cards greater than one? Jack?
Jack: No.
Janet: Are any of the decimal squares greater than one, Natalie?
Natalie: No.
Janet: How do you think we can make a whole number?
Student: Out of two halves.
Janet: Two halves. O.k.
Jack: Add two decimal cards together.

Lesson 2. The purpose of the second lesson was to introduce the concept of equivalent decimals. By looking at three symbolic representations for the same quantity, Janet challenged her students to determine whether or not they were the same or different. Also in this lesson she provided exercises designed to connect decimal and fraction representations.

Janet began by asking the class to compare various decimals represented by decimal squares on the overhead projector. In this particular example, the decimal symbols .8, .80, and .800 were offered by the children as being equivalent. Janet challenged them to support their response. The following discussion resulted.

Stdt. 1: Put them (decimal square representations) on top of each other. (Janet asked if they saw any patterns.)
Stdt. 2: Yah, adding a zero. (She wrote .8, .80, and .800, naming each one. A student responded that the pattern was times ten.)
Janet: Who said times ten? O.k., the pattern for here is times 10.

Janet then asked if anyone observed other patterns. A student suggested adding a zero as the pattern, and the discussion continued as follows:

Janet: Add a zero. So you think we could do this (she wrote .8000 on the board)? You could keep going on and on and they would equal one another? Why do you think that is so?
Jeff: They're all eight.
Janet: They're all eight?
Jeff: They all have an eight.
Janet: Yah. Do you think that's why they're equal? (Another student, Natalie, joined in.)
Natalie: They're all the same size.
Janet: Yah, they're all the same size.
Stdt.: They're all dividing by ten. (Another student suggested the decimal numbers would all reduce to the same number.)
Janet: What do they all reduce to?

Other answers (e.g., 2/10, 2/5) were suggested by the students. When one student suggested 2/5 as the correct answer, four or five others indicated agreement. Within a few seconds (enough time for the rest of the class to think about this answer) the majority of the class shouted, "No. No!" The consensus of this group was that the answer was 4/5, and the other group quickly agreed. Meanwhile children were checking their own work and the work of others.

Janet asked the class what the "one" would be if they wanted to multiply 8/10 by some form of one to get 80/100. Almost all responded, "10/10."

Janet concluded by saying "So would you agree that if we multiply these by one, we're still going to come up with an equivalent fraction like we did with the fractions?" The class responded affirmatively. She continued, "So with the decimals, even though you see a pattern of adding zeros, is it still multiplying by one to get it to the next decimal?" They again responded affirmatively. She continued with other examples.

Lesson Three. In this lesson, Janet introduced the concept of inequality, referring to earlier concepts illustrated by the decimal squares with equality. Janet asked the students to compare different symbol representations by encouraging them to imagine what the symbols meant. Her intent, it seemed, was for her students to make some connection between the symbolic and imagistic mode of representation.

In an exercise in which students were asked to compare decimals given in symbolic forms, Philip said that .03 was greater than .030. Janet asked the class, "Did everyone get that?" The class responded loudly, "No!" Janet continued.

Janet:	Now read me that problem.
Philip:	Three hundredths is less than thirty thousandths.
Janet:	Now can you picture the decimal squares, Philip? Can you picture three out of a hundred? (Philip closed his eyes.) Philip: Yah.
Janet:	What color were the hundredths?
Philip:	Green.
Janet:	Green. All right, three hundredths. Now can you picture thirty thousandths in yellow? (He closed his eyes again.)
Philip:	Yah.
Janet:	Do you think you had it wrong?

(He erased the first greater than sign and closed his eyes again.)
Philip: Equals.
Janet: Good. (Class applauded.)

By visualizing the decimal square, it seemed that Philip was able to imagine his answer to the problem. The loud dissention and supportive applause of students suggested their interest in Philip's progress and they appeared genuinely happy to see him successfully complete the problem.

Another exercise called for students to order the set of decimal numbers - 512.23, 512.023, 512.03, 512.3, and 512.32 - from least to greatest. The discussion between Janet and Paul, a student who had difficulty in generating the correct list, was as follows:

Janet: Five hundred twelve and thirty two hundredths was the biggest number there. Now think about those decimal squares we were using. We pictured thirty-two hundredths. How many rows would there be?
Paul: Three.
Janet: Three. So three whole rows and two more little squares for thirty-two hundredths. Now think about three tenths. How many whole rows?
Dan: Three.
Janet: Three. But with no rows left over. So this is just a little smaller than this (.32 and .3). (She wrote the numbers on the board in order. The students indicated that they understood.)

CONCLUSIONS

This paper is limited in that it is based on the observations of a single teacher over a six-week period. However, it documents the mathematics classroom teaching behavior of a teacher who regularly monitors the state of her students' understanding and seems to integrate this knowledge into the questions she poses and justifications she seeks. Her mode of evaluation was not through paper and pencil tests (one short quiz was given on the sixteenth day) but rather through her interactions with students. Her focus seemed to be on the assessment of meaning and understanding in children and, in the absence of appropriate instruments, invented her own.

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THE EARLY SOCIALIZATION OF MENTORED AND NON-MENTORED TEACHERS: TWO THREE-MONTH CASE STUDIES OF BEGINNING MATHEMATICS TEACHERS' PROFESSIONAL NETWORKS

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Functionalist and conflict views of socialization processes are used to interpret the experiences of two beginning teachers' evolving professional networks during the first three months on the job. The inductee in a conventional setting concluded, "I must solve my own." The second inductee was in a 60% position with a mentor. He was supported by numerous people as he sought to resolve professional conflicts during his early induction experience.

The growing recognition of the importance of the induction year for teachers has motivated schools and teacher education institutions to begin to develop induction programs for beginning teachers. Virginia Tech/Roanoke City Schools' "Mentored Induction Program in Secondary Mathematics" is an example of an effort to provide novice teachers with "mentors" or "colleague teachers" who can give support and guidance as the novices are socialized to teaching. A component of socialization of great interest is the view that individuals hold of their roles and the ways in which these views develop. Among sociologists, there are two major models of socialization - the functionalist and conflict models (Lacey, 1977). The functionalist model emphasizes the idea that socialization fits the individual to the group. The conflict model posits tensions between groups which vie for control and influence. We were particularly interested in the professional networks that novice teachers developed during their first 3 months of teaching and the ways in which the beginning teachers interacted with people in these networks. We will argue that the context in which the socialization process takes place and the composition of novice teachers' professional networks and influences the novices' views of their roles.

Two beginning secondary mathematics teachers participated in our study, Ted and Charles. Our data collection procedures included interviewing each participant twice before a full week of school-based orientation, once to gather case histories and once to get reactions to the new teachers' orientation and follow up on histories. A third interview was conducted at the end of the full week of school based orientation. During the first, fourth, seventh, and tenth weeks of teaching we observed two classes and interviewed the participants each of three days. These fifteen interviews, our observational data, and our own reflections constitute the three data sources from which this paper is constructed.

Our teacher participants were graduates of the same undergraduate teacher education program. They both taught in the same high school in a small city district. The school was a large multi-building campus of over 2000 students which employed the "school-within-a-school" concept. Ted and Charles taught in two non-adjacent buildings and rarely saw each other. Ted was in a conventional first year teaching position. On a six period schedule, he taught five classes and had a hall duty. Charles was one of two teachers in the school in a special induction program. Charles taught only three classes a day. He and a colleague, Heather, shared a position at 60% salary each. An experienced teacher, Theresa, was released from one class a day all year to serve as their mentor. All three of them had a common preparation, trigonometry, and a common planning period.

It is within this general context that the two participants began their first year of teaching. We sought to understand the personal views of the cultures of teaching and the corollary professional networks that these teachers developed as they began their teaching careers.

TED

Anticipations

Ted was asked in an interview before the beginning of the school year to anticipate who might be the most important people helping him succeed as a teacher. He responded:

> I don't know who they are yet but I'd have to say the department head in math at PHS and collectively I'd have to say the other math colleagues, people I'll be not so much working with, but working around. And probably other teachers - just from any discipline to help me get over any broad problem at work - you know - within the school or how to cope with something - just to give me their experience.

He thought the department head would have to "review lesson plans and offer feedback" and "maybe offer alternative ways of presenting the material." He thought "any discipline problem I might have I could go to almost any teacher" but suggested that experienced teachers would be the best sources of advice. However ". . .For materials or maybe a new way to bring material across in the classroom I wouldn't hesitate to go to another new teacher because, you know, they have ideas just as anyone else." He also anticipated that his wife, an accountant, would be "a sounding board" and a source of ideas for his consumer mathematics class. Students would be a source of feedback, Ted anticipated, and would let him know how he was doing.

During the initial pre-school meetings, Ted was asked by the principal to be part of an "alpha-beta" pair. An experienced English teacher, Mr. Rich, was to be Ted's "Alpha" or a teacher to whom Ted should go when he had administrative questions. Ted also met Mary John, an experienced mathematics teacher who was new to PHS and would be teaching across the hall, he thought she would be good to "commisserate with."

Realities

By early October, after four weeks of teaching, Ted had established a fairly complex professional network which remained quite stable through late November. In fact, those persons Ted had anticipated would be helpful, were helpful, in his eyes, and others had joined his network of support. He could describe this network quite explicitly and seemed very aware of the roles various people played in his teaching. The following is a summary of this network in order of importance to Ted. The persons in the network are named, their positions noted, and their roles briefly described.

Anita Fish (department head) - helped with solving a major problem with a student and suggested ways of teaching for understanding and assessing student work

Mary John and Sally Corning (other mathematics teachers in the hall) - talked with about ways of presenting material, shared hints about student difficulties

Sarah Less (history teacher in next room) - talked about teaching in general, especially about how he felt about the students he taught; shared similar ideas about how to handle students

Charles and Heather (mentored beginning mathematics teachers) - talked about how things were going, not so much suggestions as simply talking

Mr. Rich (alpha) - initially a big help with the administrative items but later was not "used" because his ideas about discipline did not appeal to Ted

Celeste (Ted's wife) - got Ted to talk about his work, listened to explanations before Ted taught them, took tests before they were given

Stella and Zack (mathematics supervisors) - provided information regarding curriculum at meetings and shared teaching ideas

Ted described this elaborate professional network rather carefully; however, he continued to talk about his teaching as though he were "in it alone" and had very little assistance. He said, for example, "There was no one really to talk with about the Beginning Teacher Assistance Program (BTAP)," although Charles and Heather also were required to participate in BTAP. He reported that he relied on himself and Mary John for teaching ideas, complaining somewhat that teachers at PHS did not share materials or ideas very much. He hypothesized that lack of a mathematics office or library may have caused this lack of sharing. He said:

> I think you're basically on your own as far as teaching your class. That's pretty much what I expected. It's your class and you have to teach it and you have to meet these SOL's and you gotta follow this curriculum, so that's laid out for ya. . . It's basically what you have to do. . . and you do it. You find a way to do it.

He said he felt like a real teacher "if being a real teacher means having a ton of problems and no answers." He reported getting different answers to the same problem from different teachers and feeling like no one really knew what to do. This may explain his feelings of being on his own.

Charles

Anticipations

Before school began, Charles thought his mentor, Theresa, would be the main professional in his environment. He also anticipated he would confer with his father, who is a relatively novice physics teacher; Gail, a high school math teacher he met in his undergraduate program; an English teacher, a school-designated helper (his alpha), and "other teachers". He expected his mentor to be like an "older sister or aunt." He said "I'd like her to always be there." He already knew that he wanted to talk with her about organizational matters. He wanted to talk with his father about "applications" and "how to handle tardies." He hoped to talk with a geometry teacher to get some ideas, and he wanted to talk with English teachers to create some language and mathematics integrated learning experiences for his students.

Early Networking

During the two weeks of orientation and first week of classes, his interview comments centered about his mentor (20 instances), other school faculty (16 instances), people in the induction program (7 instances), his induction partner, Heather, (8 instances), the building level administration or dean (4 instances), and the math ed faculty (5 instances). More specifically, Charles had already met with the mentor, Theresa, before classes started and they were setting up daily meetings during their common planning period. They "discussed points of view;" "how to teach math";" devised guidelines "for running classes, grading, testing, homework, quizzes, notebooks; and developed a letter to parents. Theresa made several specific suggestions on how to solve the problems he would have "floating" from room to room and had already cautioned him to "slow down." Theresa shared insights about students and pedagogy, especially related to trigonometry, their common course.

He was getting ideas from two other participants in the induction program with whom he met in weekly seminars and, through them, ideas from their mentor. He also mentioned the value of the conversations, observations and reflection sessions conducted by the university faculty members.

Charles met with Heather and Theresa frequently. Heather and Charles also spent the first, fourth and lunch periods together daily and talked in the evenings. Charles said of her, she's "been a great influence. She's shown different ways of presenting things. (She) has been very, very supportive." Charles also felt he had "great support" from the building dean. Matters were quickly clarified about Charles' role and the dean shared written information on how to handle discipline problems. Charles was feeling some stress about an emminent observation by the Dean. He felt pressure to perform well for the observation.

There was a new department head so the department had only a brief meeting for introductions. Charles had met several math teachers and chatted for "five or ten minutes" with several of them. He had delivered math texts with Ted but had little additional contact with him. Charles thought his colleagues were "real nice." He had met a geometry teacher across the hall, but with her six period day filled with five classes and a duty, there was no time to talk with her even though he taught only three periods.

The Third Month

In interviews conducted during the seventh and tenth weeks of classes, there was a dramatic increase in the number of instances in which he referred to Theresa, his mentor, (30 times) and Heather, his induction partner, (27 times). He also mentioned, but much less frequently, contacts made through the induction program, parents, other math faculty in the school, contacts through professional development activities, and non-math faculty.

The basic pattern of regular and frequent interactions occurred with Theresa and Heather. They talked daily and met twice a week or so to discuss teaching, especially trigonometry. Charles began to assert independent thinking; "I deviated a little bit differently than the way (the mentor) did it." He thought his mentor was a good teacher, but he saw her as having a much more structured approach than he. He saw himself as more of an "actor" and didn't mind student talking if learning was occurring. He trusted the mentor's expertise; "If I don't agree with what (she's) doing, I mean I feel kinda worried about it. Cause, I mean, she's been teaching for a long time."

Sometimes he talked with Heather. He felt better if they agreed. Charles had also observed Heather teach and vice-versa. He found the experience valuable. They talked about the observations. They also talked about their state evaluation visits. Mostly Charles and Heather interacted many times daily exchanging ideas and talking about small problems. Heather was "just a friend, I guess, and I talk to her a lot. I talk to her like at night, sometimes on the weekend... she's also a teacher. And she can help me in what I'm doing." He also talked with a teacher across the hall who taught geometry and asked her questions several times a week. He continued to get ideas from and to reflect with members of the induction program, students and faculty.

DISCUSSION

Both Ted and Charles faced the typical set of problems of beginning teachers (Veenman, 1984) and had similar anticipations of the networks they would develop during their first year of teaching. Both had hoped to draw on the wisdom and expertise of the people around them to help them succeed. The contexts in which they attempted to develop these networks were similar in many ways and yet very different. Tim, because of his full schedule and isolation, had to work very hard to establish communication with others. He had to seek out others, very little was offered to him. The advice he received and the experiences he had led him to believe that it was his responsibility to accommodate himself to the local school culture, a rather functionalist view of socialization.

Charles, on the other hand, was a participant in a program which was designed to offer him assistance during his first year of teaching. His mentor had many of the desirable features of a mentor (Gray & Gray, 1985) and Heather and others affiliated with the program were very available to him. He also sought out others for advice. Charles seemed to view his role as one of negotiating to do what he believed was best in his classroom, a conflict view of the socialization process. There were many challenges, but his mentor and the program faculty and participants helped him negotiate conflict resolutions in ways which helped him maintain a strong personal sense of growth and control over his professional life.

Although Ted and Charles were very different in personality and outlook, we believe that each of them would have had very different induction experience if the contexts in which they were working had been reversed.

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EXPLICIT KNOWLEDGE OF INFINITY

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The explicit and implicit knowledge of infinity of 70 teachers was assessed using a ten-item instrument. Less than half of the subjects were judged to have an adequate knowledge of the explicit symbolization of infinity. Many subjects were inconsistent when their explicit knowledge of infinity was contrasted with their responses to tasks involving implicit knowledge of infinity.

Introduction. Knowledge of mathematical infinity among school children and teachers is a complex mix of schooled and unschooled knowledge, and knowledge that may be counter to intuition, context-sensitive, extensive yet incomplete, and naive yet sophisticated. Without the intervention of formal instruction, most preschool and primary children recognize that there is no largest number (Evans, 1984). Fishbein, Tirosh, and Hess (1979) found children from age eleven have extensive, intuitions of infinity in geometric settings. Most infinity research focuses on pre-college populations. An exception is the Martin and Wheeler (1987) study which found preservice elementary school teachers to hold incomplete and inconsistent concepts of infinity on tasks which minimized the impact of explicit schooled knowledge. This study looks at preservice elementary school teachers but focuses on their explicit knowledge of infinity. Because conceptual knowledge does not consist of discrete pieces of information, relationships between explicit and implicit concepts of infinity were considered. The questions addressed were

1. What explicit, declarative knowledge of mathematical infinity is held by teachers?
2. How are explicit and implicit knowledge of infinity related?

Method. To answer these questions, a ten-item open-ended instrument was developed. Six items (items 1, 2, 3, 5, 6 and 7) focused on explicit knowledge of infinity and its symbolization. Four items were paper-and-pencil adaptations Martin and Wheeler tasks (1987), concerning implicit, intuitive concepts of infinity. Six items (4, 6, 8, 9 10) required a forced choice (true-false; yes-no) and an explanation. Items were presented on separate sheets of paper. Subjects provided written explanations on the same sheets without recourse to previous responses. Test administration was 20 minutes. Subjects were elementary school preservice teachers enrolled in an upper-division course in mathematics methods at a large university.

Results. Results for explicit and implicit knowledge of infinity are reported separately. When appropriate items are summarized as clusters of items. Subsequently implicit and explicit results are contrasted.

Explicit knowledge results. The first three items focused on standard symbols commonly associated with mathematical infinity. These items allowed assessment of familiarity with syntactic conventions.

> Item 1. Explain the symbol ". . ." as you might to a fourth-grade student when it appears in the following mathematical sentence: 1, 5, 25, 125, 625, . . .
>
> Item 2. Explain this symbol as you might to a fifth-grade student: ∞
>
> Item 3. Explain this symbol as you might to a teaching colleague: ∞

Item responses were judged adequate if phrases such as "infinite", "continuing on and on" or "there is no end" were used. No individual item received 60% or more of the adequate responses. Important differences between the explanations for items 2 and 3 were not found. For each subject, the collective responses were judged adequate if two or three of the individual responses were adequate. Summatively, 44% of the subjects provided adequate responses. Printed symbolism typically associated with infinity was unfamiliar for more than half of the subjects.

Items 5, 6 and 7 focused on infinity as explicit vocabulary.

Item 5. What is infinity? Explain as completely as possible.
Item 6. Is infinity a number? 'yes 'no because . . .
Item 7. What is infinity plus one? Explain.

Item 5 responses emphasized two processes: a recursive process incorporating the idea of successor and an unending, ongoing process. Subjects responding with a successor definition wrote:

> S_{113} "Infinity means . . . no matter what number you say, there is always one greater simply by adding one to it."

Respondents in the unending process category typically wrote "Infinity means the numbers go on without stopping".

As shown in Table 1, six (9%) subjects concurrently acknowledged infinity to be an iterative process and not a number. Among 63 responses identifying infinity as a process only two utilized a geometrical context.

Table 1.
Response Frequencies for Items 5 and 6

Item 5	Item 6	
	yes	no
successor	3	6
unending	11	43
unclassifiable	2	5

For item 7 about as many subjects argued infinity plus one to be "infinity" (N = 23) as argued it to be meaningless or impossible (N = 22). Typical explanations from each classification follow:

> S_{221} "For every number we can think of, name, and count, we can think of, name, and count one more."

> S_{105} "Infinity means going on forever, so infinity plus one has no meaning."

With a single exception among these 45 responses, respondents also checked "no" to item 4, an implicit item concerning the existence of a largest number. Another 16 subjects explained "infinity plus one" in terms of whole number arithmetic.

Implicit knowledge results. Four items measured implicit knowledge of infinity. Individual item results are reported.

Item 4. Is there a largest number? 'yes 'no because . . .

More subjects answered this item correctly than any other item. Sixty-four (91%) subjects responded no; 6, yes. About as many subjects (N = 29) explained the absence of a largest number using the add-one algorithm as argued the existence of an infinite number of numbers (N = 26). Another 12 subjects said "no" and explained that new numbers were always being discovered.

Item 8. TRUE or FALSE: Every line segment contains an infinite number of points. 'true 'false because . . .

At the forced choice level, item 8 results are as follows: true, 39; false, 24; no response, 7. Twenty-nine of the 39 true explanations were classified into three categories: procedural, 13; existence, 4; counting, 12. These responses are "acceptable" in table 2. A sample of each follows:

> S_{101} "True . . . because . . . no matter how small we choose to divide up a line segment there will be an even smaller division that could be conceived."

> S_{310} "True . . . because . . . although limitations may hinder a visual concrete example, there are a never ending amount of points."

> S_{308} "True . . . because . . . we could never count all the points on a given segment of a line."

> Item 9. Consider the set of WHOLE numbers and the set of EVEN numbers. Check each of the TRUE statements. [there may be more than one; there may be none].
>
> ☐1. Both sets have an infinite number of elements.
>
> ☐2. The sets are "equally numerous".
>
> ☐3. The WHOLES are "more numerous" than the EVENS.
>
> EXPLAIN how you decided whether each statement was true or false.

Most subjects (80%) responded that the sets of even and whole numbers have an infinite number of elements. One-third of the subjects said these sets were "equally numerous"; more than half did not agree that the set of whole numbers is "more numerous" (56%). Few subjects (29%) argued the sets of evens and wholes to have an infinite number of elements and to be "equally numerous". These 20 responses are classified "acceptable" in table 2.

> Item 10. TRUE or FALSE: There exists a smallest fraction greater than zero. ☐true ☐false because . . .

Subjects responded to item 10 as follows: false, 29; true, 28; no response, 13. Sixteen of the false responses were classified as recursive explanations. These explanations were in two categories: procedural (11) and an existence explanation (5). Representative explanations include the following:

> S_{103} "False . . . because . . . no matter how small the fraction, the denominator can be made larger by one which makes an even smaller fraction."

> S_{309} "False . . . because . . . we don't know the name of the smallest fraction."

Among the remaining 13 false responses, eight focused on zero and/or negative numbers. Representative responses include the following:

> S_{213} "False . . . because . . . whole numbers are greater than zero; fractions are less than zero."

> S_{316} "False . . . because . . . the smaller fractions other than zero would be in the negatives and therefore would be smaller than zero."

Most of the 28 true reponses (82%) argued the necessity of a smallest fraction (7) and/or argued from the premise that zero is nothing (16). Representative explanations include the following:

> S_{206} "True . . . because . . . you have to start somewhere and no matter how small the fraction there are only so many numbers between zero and one."

> S_{108} "True . . . because . . . zero is nothing so the smallest fraction greater than zero is the first bit of something."

In table 2, false responses are acceptable; true responses, unacceptable.

Discussion. Conceptual knowledge can be characterized as knowledge that is rich in relationships--a network for which the the discrete pieces of knowledge are just as important as the linking relationships (Hiebert, 1986). Contrasts in table 2 identify weaknesses of linking relationships among these subjects. Teacher candidates are inconsistent when their explicit knowledge of infinity is contrasted with their implicit knowledge. For each contrast summarized in table 2, the number of responses classified as acceptable-unacceptable exceeds those classified as acceptable- acceptable or unacceptable-acceptable.

This infinity research contributes to an understanding of one part of the student-teacher-instrucion triad--the teacher. Subsequent research needs to be built on student-teacher interaction research and on the knowledge of particular student groups and the assigned teacher rather than to consider students and teachers as separate entities.

Table 2

Explicit items contrasted with implicit items

Explicit Knowledge Implicit Knowledge

There exists a largest number (item 4)

	Acceptable	Unacceptable	Unclassifiable
Acceptable	29	2	0
Unacceptable	35	4	0

Line segment has an infinite number of points (item 8)

	Acceptable	Unacceptable	Unclassifiable
Acceptable	15	10	6
Unacceptable	24	14	1

Contrast the even and odd numbers (item 9)

	Acceptable	Unacceptable	Unclassifiable
Acceptable	7	21	3
Unacceptable	13	26	0

There exists a smallest fraction greater than zero (item 10)

	Acceptable	Unacceptable	Unclassifiable
Acceptable	14	10	7
Unacceptable	15	18	6

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TEACHER AS LEARNER; LEARNER AS TEACHER[1]

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The paper is a case study of a teacher who participated in a unique continuing education program for teachers in one school district. The program, the Teacher Study Group on Mathematics, provided opportunity for teachers to develop their understanding of mathematics through doing and discussing math problems. The paper examines the teacher's participation in the Study Group and ways in which her experiences were transformed in her class.

Introduction

One limitation of mathematics instruction in today's schools is that mathematics is taught as a body of knowledge to be absorbed rather than experienced. The emphasis in the classroom is mastery of procedure rather than development of understanding (Romberg & Carpenter, 1986). Bishop (1985) proposed that mathematics teaching be viewed as controlling the organization and dynamics of the classroom for the purpose of sharing and developing mathematical meaning. Bishop defined this as knowledge which connects with the individual's current knowledge about mathematics, other subjects, and about the real world. In this paper, we look at Helga, a first grade teacher in a small midwestern school district, whose participation in a teacher study group on mathematics helped her create an environment in her class which approaches Bishop's model of mathematics teaching. The paper focuses on Helga's adaptation of one activity from the study group for use with her students, developing it to facilitate study group for use with her students, developing it to facilitate communication and thinking about mathematics. One author

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observed Helga during meetings of the Teacher Study Group on Mathematics, and the other observed Helga's mathematics teaching. Data consist of fieldnotes and transcripts of audio recordings of Study Group and classroom observations and transcripts of three interviews with Helga.

TEACHER AS LEARNER: THE TEACHER STUDY GROUP ON MATHEMATICS

Magdalene Lampert, a professor of Teacher Education at a near-by university and an elementary math teacher, organized the Teacher Study Group on Mathematics (TSGM) as a continuing education program for teachers of Helga's district. The TSGM was based on a belief that teachers need to develop understanding of mathematics through doing and discussing math problems. The teachers talked about mathematical content and processes, discussing what these meant to them and their students. They considered different ways of knowing and understanding mathematics (Lampert, 1988; Parker, 1988).

For most of the hour and a half weekly meetings Lampert presented one problem, which could be approached in different ways. Teachers worked on problems individually for about ten minutes. Sometimes pairs explained their strategies to each other. Then the whole group discussed the problem, knowledge they brought to bear on it, and strategies for solving it. The director helped participants explain and expand their thinking by asking such questions as: Where did that number come from? Why is that relevant? Was there any other way you could have arrived at that? She did not take the role of sole mathematical authority. Participants learned to evaluate reasonableness of strategies and answers, justifying answers by checking them against problem constraints. They revised their thinking as they interacted with other participants. Through discussion participants developed an understanding of multiple representations and solutions to math problems. They moved from viewing mathematics as a static, rule bound discipline in which problems had one right answer to viewing it as a dynamic process in which answers were mutually constructed and socially ratified (Lampert, 1988).

During several meetings, teachers talked about previous experiences with mathematics, what the TSGM meant to them, and changes in their thinking about math. Helga said she learned more than just mathematical content through participation in the group:

> While you're giving us more information about mathematics content, you're also modeling a teaching style. You're also modeling use of group process. You're modeling... a lot of things.

Helga also learned a new way to think about knowing and doing math. She began to think about math teaching and learning as social processes so her teaching centered on thinking about math and making sense of it, rather than learning correct procedures and right answers.

LEARNER AS TEACHER: TRANSLATION TO CLASSROOM PRACTICE

In the second TSGM meeting teachers worked the egg basket problem:

> There are fewer than six dozen eggs in a basket. If I count them two at a time, there is one left over. If I count them three at a time, there are none left over. If I count them 4, 5, or 6 at a time, there are always three left over. How many eggs are there?

Helga sought a way to adapt this problem for use with her first graders.

She chose the activity to motivate students and help them "understand numbers better". She used the activity for several weeks, structuring it to reflect her TSGM experiences. Each week Helga chose a "mystery number," giving students one clue a day. Clues were presented on four successive days so that each day's choices narrowed the range of possibilities, revealing the mystery number on the last day. Helga wrote the clue on a chart, which included previous days' clues and possibilities. Students worked in groups or individually to list their choices in dittoed books. The class then discussed those choices. Helga called on students to justify their choices and explain strategies to make choices. The class agreed which numbers fit the clue, then corrected choices listed in their books.

Helga guided student discussion, helping children talk about the reasoning and strategies. She explained why discussion was an important part of her math teaching:

Figure 1: One Week's Mystery Number Clues

Monday: The mystery number is between 63 and 82.
Tuesday: You will say the mystery number if you count by 5's.
Wednesday: If you have the same number of cookies as the mytery's number, ten children could have the same number of cookies and it would be fair.
Thursday: You can show the mystery number on the mini-computer[1] with one checker.

[1] A manipulative device the children used in their regular curriculum

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> My strategy that I come up with and then I share with the children might not work, but if they share, maybe six different strategies will come up. There might be a strategy that will make sense for a child, that they might also then be able to apply in a different situation. So it's like sharing your thinking with each other.

Helga asked students to expand on their answers, explain strategies and justify choices. One day's clue was:

> There are five children who want cookies. Pretend they have as many cookies as the mystery number. They will each have the same number of cookies.

The choices left were 13 and 15. Helga started the discussion by calling on Tony:

Tony: I think it's 15 because of 5, 5, 5.
Ned: I think it's 15, too, because we say five when we count 5, 10, 15, 20.
Helga: Brent, what's your idea?
Brent: I was thinking ahead. Every---If five children have three, I used my fingers and found I ended up with 15 counting by three.

Helga helped students articulate what they meant and explain it to others. She also helped children articulate what they found confusing or did not understand. In the following example of discourse, the clue for the day was:

> Pretend you have the same number of cookies as the mystery number. Ten children could have the same number of cookies and it would be fair.

The possibilities were 65, 70, 75, and 80. The students and Helga had demonstrated that 70 and 80 were possibilities in a drawing in which the "cookies" were distributed one at a time to ten "children." She asked students if they found the lesson confusing or didn't understand it. Beth raised her hand, and Helga asked what she found confusing:

Beth: I didn't know how to give--the number to give to each child.
Helga: Okay. You didn't know how to start. Boys and girls, look this way. Let's start with the ten children. Beth has a good question. She said, "How do you know how many to start giving?"

Helga, rather than explaining it herself, called on another student to explain his thinking.

Brian: Give each -- start out by giving everybody one cookie.
Helga: And so you started out by giving everybody one. How many

did you use up?
Brian: Ten.
Helga: That's not enough, is it? 'Cause I know there's 80. So give everybody another one.

Helga moved away from being sole classroom authority. She asked students to justify choices by checking them against the constraints of the clue, rather than asking her to ratify their answers. She encouraged students to leave some questions open, moving away from a belief that mathematical questions have one right answer. Through discussion Helga moved students' visions of math as reified and external to a view of math as something experienced with others and constructed in social interaction. Helga revealed her concern with discussion as a means to develop understanding and the nature of mathematical authority during review of the previous day's class discussion, when the clue was "The mystery number is less than the number of children in the class."

> I was very proud of your good discussion and your good thinking, especially in our discussion of whether zero has been included in our list. Some of you felt that zero should have been included and some of you felt that zero should not have been. I left it up to you to decide what you wanted to do.

Rather than directly confirm or reject a student answer as wrong, Helga had students check their answers against clue constraints or turn to other students to sanction answers. In this example, the clue was:

> "Pretend you have the same number of cookies as the mystery number. Ten children would each have the same number of cookies and it would be fair."

The choices were 65, 70, 75 and 80.. The class agreed that 70 was possible by distributing seven to each child. Helga asked Ed if 80 was a possibility:

Ed: Uh-uh. It wouldn't be fair.
Helga: Why not, Edward?
Ed: If you put eighty, some children wouldn't get as many.
Helga: Rob, do you have an idea?

Rob: I think 80.
Helga: You think 80. How many do I give each child? They have 7, so how many more do I add?
Rob: One.

Helga incorporated student contributions (sometimes unsolicited) into her lesson and built on them. In one instance, students decided the Mystery Number was 80. Rather than saying "Correct" and ending the lesson, Helga continued. She asked Pel to show 80 on the mini-computer. After representing 80 the standard way, Pel offered two other ways. Helga then involved the students in a discussion of various ways to represent 80. Showing 80 in various ways took several minutes. But a student felt it was important and involved teacher and peers in his excitement for doing math. We saw many instances when Helga incorporated student contributions into instruction. In one TSOM meeting Helga talked about letting students hold the floor:

> My objective for (the Mystery Number activity) is to have discussion in thinking about math. It's not just to teach addition or whatever...What I think the kids like about (the activity) is I'm letting them take off with some of their ideas.

In another meeting Helga talked of her students' changing perception of the nature of mathematics during several weeks of the Mystery Number activity:

> Now the kids are not just looking for the right answer. They're looking for ways -- different ways of getting an answer. And they're also realizing they don't need the teacher to say what's right. They can figure out that they've done it correctly or not. And that's very rewarding.

We believe that with the creation of the Mystery Number, Helga controlled the organization and dynamics of the classroom, as Bishop (1985) suggested. Students engaged in sharing and developing mathematical meaning, developing a view of mathematics as social and dynamic.

CONSTRUCTING MATHEMATICAL MEANING

What Helga and her students experienced in class paralleled what she and her colleagues experienced in the Study Group. Participants in the TSGM developed more sophisticated and complex ways of thinking about mathematics through engaging in mathematical discourse. They had opportunity to reflect upon their learning and their way of making sense of problems. Teachers were seeing themselves as mathematically competent and no longer dependent on the authority of others (Parker, 1988). Helga's work on a TSGM problem led to creation of the Mystery Number activity. Continued development of the Mystery Number activity, with Helga's changes in the traditional structuring of authority and her emphasis on discussion and thinking about math, reflected her growing understanding of mathematics. In the TSGM Helga, "teacher as learner," participated in construction of a new understanding of mathematics. In her class Helga, "learner as teacher," created an environment in which her students could participate in construction of mathematical meaning. Our study of Helga illustrates one way a teacher was able to foster a sense of mathematical competence in her students.

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The Influence of Symbolic/Graphic Calculators on the Perceptions of Students and Teachers toward Mathematics

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This paper describes the results of a study of secondary students' and teachers' attitudes toward calculator use and related perceptions and beliefs about mathematics. Students in Algebra II, Trigonometry, and Calculus classes at two high schools were assessed with multiple Likert-type attitude scales both before and after six months experience with a powerful hand-held programmable symbolic/graphic calculator (HP-28). Factor-analytic results suggested that male and female students' attitudes changed in very different ways, with "utility" and "enjoyment" being the principal concerns of male and female students, respectively. Interviews, surveys, and classroom observations of teachers who used the calculators indicated that the calculators gave teachers opportunities to try different teaching approaches that they would not have had otherwise.

INTRODUCTION

The uses of powerful technology in the mathematics classroom continue to expand and develop. Two broad classes of computer software - interactive graphic function plotters and symbolic algebraic manipulation packages - have stirred considerable interest in mathematics education for the possible impact they could have on how students learn and how teachers teach mathematics. Several projects have utilized such software in college calculus and precalculus mathematics courses (Heid, 1988; Lane, Ollongren, & Stoutemyer, 1985).

An overriding limitation on the potential for widespread use of this technology in the secondary classroom has been the problem of access. We have seen symbolic/graphic capabilities which were once the exclusive province of large main-frame computers become readily available on personal computers. Still, few schools can boast of such abundant economic resources that every mathematics student could have ready access to a personal computer during every mathematics class. A separate computer lab may be available for occasional use by mathematics students (as well as programming and business students). A more likely scenario has at most one or two computers available to the mathematics teacher for classroom demonstrations. And in virtually all cases, the variety or unavailability of home

The authors would like to express thanks to Joan M. Shaughnessy, who designed the Likert scales and performed much of the statistical analyses of the student data.

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computers makes teachers justifiably hesitant to assign homework requiring the use of computers.

This question of access to graphics and symbolic algebra packages is rapidly becoming moot, as several companies have recently introduced graphic scientific calculators, and the Hewlett Packard Corporation has developed an extremely sophisticated hand-held calculator (HP-28 model) capable of

1) performing calculations with symbolic expressions (including expansions of powers of polynomials, symbolic solutions of linear and quadratic equations, symbolic differentiation and the computation of Taylor polynomials for functions);
2) computing with complex numbers, vectors, and matrices;
3) plotting graphs of functions and relations;
4) finding roots and definite integrals of functions with powerful numerical methods.

While certainly expensive by calculator standards (the HP-28 retails for $235), entire classroom sets of these symbolic/graphic calculators could still be purchased for the price of two or three personal computers equipped with the equivalent software.

THE STUDY

With the help of funds granted through Oregon's Title II Program and with the help of the Hewlett Packard Corporation, two high school mathematics departments were each provided with one classroom set (30) of the HP-28 symbolic/graphic calculators. This provided an classroom environment where every student had individual access to powerful graphic and symbolic algebra capabilities on a regular basis. The five teachers involved in the project used the calculators in Intermediate Algebra, Trigonometry, and Calculus classes. Each teacher used the calculators in two different subjects over a semester's time. These teachers were encouraged to consider alternative approaches and instructional strategies for traditional topics in the high school mathematics curriculum, and to consider new topics which heretofore would have been inappropriate without such technology available.

The availability of this technology in these classrooms raises several questions:

*How do students' attitudes and beliefs about mathematics and calculators change as a result of using such powerful technology?

*How do the dynamics of classroom interaction change in this environment?

*What are the teachers' perceptions of the proper role and use of such technology in the teaching of mathematics?

•In particular, what types of mathematical tasks in the usual curriculum are enhanced (from both the student's and teacher's viewpoints) and what new tasks do teachers attempt that they might not have tried without the calculators?

This study attempts to provide some initial answers to these questions.

METHOD

At the beginning of the project, 294 students (51% male, 49% female) in the experimental classes were administered multiple Likert-type attitude scales (25 items total) designed to assess student attitudes and beliefs about mathematics and calculator usage in general. Students in the Algebra and Trigonometry classes comprised 92% of the sample and had not yet seen the HP-28 calculators at the time of assessment. Some students in the Calculus classes had been briefly introduced to the HP-28 calculators and six already owned one. Over 90% of all students in the sample reported having some type of calculator at home and over 90% reported use of a calculator on either a daily basis or a few times each week.

At the end of one semester's experience with the calculators, the experimental classes were again administered the same attitude scales (251 students). The researchers made classroom observations, conducted open-response surveys of teachers and students regarding pros and cons of using the calculator, and conducted personal interviews with the teachers regarding their own perceptions of students' use of the calculators, changes in classroom interaction (both student-student and teacher-student), and their own feelings regarding the proper role of the calculator and attitude changes in students.

Pre- and post-test comparisons (t-test with .05 level of significance) were made of individual attitude scale items. Item comparisons were also made by sex and by class. Factor analyses of the attitude scales were performed on both pre- and post-test results by sex.

RESULTS AND DISCUSSION

Changes in student attitudes

For all classes using the calculators, there was a significant difference on the following items at the end of a semester's experience with the symbolic/graphic calculator:

MORE AGREEMENT AFTER EXPERIMENT

- It is a real challenge to work problems with a fancy calculator.
- Working with a calculator helps me learn more about difficult math topics.
- Using a calculator makes learning mathematics more fun.

LESS AGREEMENT AFTER EXPERIMENT

- Expensive calculators look too complicated for me.
- There are many different ways to solve most mathematical problems.

The changes in the responses to the calculator items are not too surprising, but the change in the response to the problem-solving item is intriguing. Whether such a shift is natural with continued mathematical study or has some connection to the use of the calculator deserves further attention. In an open-response survey administered at the end of the project, students were asked what was most difficult about using the symbolic/graphic calculator. The overwhelming majority responded that remembering key sequences and proper syntax for entering information were the most frustrating aspects of using the calculator. Whether the syntactic rigidity of the calculator influences a student's overall view of mathematics in general bears more investigation.

Course differences

The most striking (and reassuring?) differences in attitude between classes (both pre- and post-tests) were between the Algebra II and Calculus students:

STATEMENTS ALGEBRA II STUDENTS AGREED WITH MOST STRONGLY

- It is a real challenge to work problems with a fancy calculator.
- Memorizing formulas is the most important part of mathematics.
- Calculators solve problems better than people do.
- I'll never get the hang of working hard problems with a calculator.

STATEMENTS CALCULUS STUDENTS DISAGREED WITH MOST STRONGLY

- It is a real challenge to work problems with a fancy calculator.
- Memorizing formulas is the most important part of mathematics.
- Calculators solve problems better than people do.
- I'll never get the hang of working hard problems with a calculator.

The Trigonometry classes agreed more strongly than the Algebra II and Calculus classes with these statements:

- Calculators are great to work with.
- Using a calculator makes learning mathematics more fun.
- Working with a calculator helps me learn more about difficult math topics.

Sex differences

At the end of the project there were significant differences between male and female students on the following items:

MORE AGREEMENT/MALE STUDENTS

- To do well in mathematics, you have to learn to follow a set of strict rules precisely.
- I feel comfortable with all sorts of calculators.
- Memorizing formulas is the most important part of mathematics.
- Anyone can do well in mathematics with enough work.

MORE AGREEMENT/FEMALE STUDENTS

- It is a real challenge to work problems with a fancy calculator.
- Expensive calculators look too complicated for me.

These differences suggest a greater tendency among male students to view mathematics as a rule-dominated discipline and a greater anxiety on the part of female students toward using calculators. This interpretation seems supported by further analysis of sex differences in how attitudes changed from the beginning to the end of the project. A factor analysis (orthogonal transformation, varimax) was performed separately for the pre- and post-test attitude scales by sex. The dominant factors which emerged from this analysis reveal sex differences which could not be discerned from individual item analysis:

ITEMS WITH HEAVY LOADINGS IN MOST INFLUENTIAL FACTOR FOR MALE STUDENTS

Before Calculator Experience

- To work with a complicated calculator a person has to be really smart in mathematics.
- Calculators are great to work with.
- Some problems are easier to solve without a calculator. (negative loading)
- When I use a calculator, I am more likely to enjoy working on math problems.

After Calculator Experience

- I feel more sure of my work when I use a calculator.
- When I use a calculator I am more likely to enjoy working on math problems.
- Using a calculator makes solving a problem more complicated. (neg. loading)
- Calculators make math problems easier.

ITEMS WITH HEAVY LOADINGS IN MOST INFLUENTIAL FACTOR FOR FEMALE STUDENTS

Before Calculator Experience

- I feel like I am cheating when I use a calculator.
- Using a calculator makes solving a problem more complicated.
- Using a calculator takes the creativity out of solving math problems.
- Some problems are easier to solve without a calculator.

After Calculator Experience

- When I use a calculator I am more likely to enjoy working on math problems.
- Calculators make math problems easier.
- Using a calculator makes learning mathematics more fun.
- Working with a calculator is the best part of my math class.

The utility and power of the calculator seems to be of primary concern to the male students, and agreement with the statement "When I use a calculator I am more

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likely to enjoy working on math problems" seems likely to be a reflection of male students' recognition of that utility and power. On the other hand, the most influential factor for female students prior to the calculator experiment seems to be a "rationalization" factor for anxiety toward calculator use (the second most influential factor in the pre-test analysis loaded negatively on "I feel comfortable using all sorts of calculators" and positively on "Expensive calculators look too complicated for me" and "To work with a complicated calculator, a person has to be really smart in mathematics"). The post-test shift to a factor reflecting the relationship of the calculator to fun and enjoyment in mathematics for female students is encouraging, and certainly suggests further study for clarification of its significance.

Perceptions of teachers

The five teachers involved in the project were self-selected, so they all understandably were favorably inclined toward using the symbolic/graphic calculator in the classroom. The teachers estimated that they each used the calculators in class approximately 10-20% of the time, and that in the future they would expand that usage to 20-25% of total classtime. The teachers developed many of their own student materials for use with the calculator (one of the original objectives of the project), and used the calculator with their conventional text materials as well.

It is safe to say that all the teachers were more impressed with the graphic capabilities of the calculator than the symbolic capabilites, and all agreed that the teaching of graphing was enhanced in all classes by the use of the calculator. Indeed, four of the five teachers reported taking much more of an "exploratory" approach to teaching graphing topics than they ever had before. For example, we observed one teacher have his students graph fifteen different examples of exponential functions in approximately twenty minutes. The rest of the classtime was spent with the students drawing (and testing) conclusions and generalizations based on patterns they discovered in those examples. Other topics which teachers felt were particularly enhanced by the use of the calculator were treatments of complex number arithmetic, matrix computations, definite integration, and evaluation of expressions with "messy" numbers. The teachers noted that in general, more complicated examples and more depth of coverage of many topics were made possible by the calculator.

The symbolic capabilities of the calculator were felt to be of less advantage, particularly for dealing with simple algebraic expressions and equations in Algebra II. Two salient reasons for this were essentially related to input/output. First, if the time to enter the necessary keystrokes approached or exceeded the time needed to solve a problem mentally or with written work, then the calculator was not deemed useful. Graphing "by hand" can be very time-consuming, so the benefits of the calculator were immediately felt for graphing applications. However, just as with simple numerical calculations, simple algebraic manipulations are more efficiently done mentally or with written work.

Secondly, the calculator accepted and displayed algebraic expressions with a notation similar to BASIC (for example: $x^2 + 2x$ would be entered and displayed as X^2+2*X). Students found little difficulty in learning to *enter* algebraic expressions with this notation, but time was often lost in the reading and "translating" of the display. It would have been most helpful for the calculator to display algebraic expressions in conventional notation, and this is a matter which should be addressed by symbolic calculator manufacturers.

The teachers felt that the use of the calculators brought only minor changes in the dynamics of classroom interaction. More student questions related to the calculator were noted on the days it was used, and students tended to help each other with calculator questions. All the teachers indicated that a few students had enough difficulty with the calculator that its use did not make a positive difference in their learning. The teachers' own perception of student attitudes was that the calculator was viewed quite favorably by the vast majority of students, and at least two of the teachers reported of an individual student whom they felt changed her plans in favor of enrolling in future mathematics primarily because of the calculator experience.

In conclusion, this study indicates that there is ample evidence to suggest that student attitudes toward calculators and related perceptions and beliefs about mathematics can be affected by the use of technologically powerful tools such as symbolic/graphic calculators, and that these changes may differ for male and female students. The meaningful *new* pedagogical opportunities provided teachers by these tools is very promising.

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A MATHEMATICS LEARNING STYLE INSTRUMENT SENSITIVE TO GENDER DIFFERENCES IN SENIOR HIGH SCHOOL AND COLLEGE STUDENTS

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Abstract. The 150-item Mathematics Learning Style Profile (MLSP) was administered to 527 senior high school and college students. The following factors were identified: Mathematical Sophistication, Perceived Teaching/Learning Style Match, Technical Orientation, Nonstandard Approach to Problem Solving, Visual Preference, and Desire for Structure. Gender-related differences were found in the response patterns for three of the six factors.

During the past decade, a number of instruments assessing learning styles have appeared (e.g., Dunn, Dunn, & Price, 1978; Keefe & Monk, 1986; Letteri, 1980). The majority of the assessments, however, are not subject-specific, focusing instead on underlying cognitive constructs which can affect how an individual typically approaches any learning task. The instrument described in the present paper, the Mathematics Learning Style Profile (MLSP), was specifically developed to assess the experiences, attitudes, learning habits and preferences, and interests of young adults enrolled in Algebra II. The present study was conducted to isolate and identify the primary factors underlying the MLSP, to compute the reliabilities of each, and to explore the possibility of gender differences in response patterns on the MLSP. It was hoped that any differences which might be found would help shed some light on the disparity in the numbers of males and females selecting mathematics-related majors in post-secondary institutions.

METHOD

Subjects. Four hundred twenty-one Algebra II students from six high schools in the Columbus, Ohio, city school district and 106 undergraduates enrolled in an Algebra II-level mathematics course at The Ohio State University (OSU) served as subjects. Each of the latter group had previously taken Algebra I, Geometry, and Algebra II in high school. The final sample consisted of the 397 high school and 105 university students who completed the learning style instrument. The mean age of this sample, which consisted of 234 males and 268 females, was 18.5 years. Race was not assessed, but members of minority populations (Black, Hispanic, or Oriental) comprised approximately 20% of the sample.

Instrument. The preliminary version of the Mathematics Learning Style Profile contained 150 items, 120 of which were designed to assess student characteristics relevant to the learning of mathematics. Questions such as, "I used to enjoy taking things apart to see what made them work," and "I question a lot of things that other people just seem to accept," were included since such experiences or behaviors may be related to the way one approaches math problems. The remaining 30 items assessed the subject's hand and eye preferences and the handedness of the subject's siblings, parents, and grandparents. (Due to space limitations, this information is not included in the present paper. These findings, along with correlations of the subscales to grades in Algebra I, Geometry, and Algebra II, will be described in later papers.)

Procedure. The MLSP was administered to the high school students during late May and early June, 1986. The college sample was surveyed during February and March, 1987. Each student received a copy of the MLSP, a #2 pencil, and a computer scan sheet on which to record their responses to the items on the MLSP. Responses on the first 120 items could range from (A) "Strongly agree" or "This statement is ALWAYS true or ALWAYS describes me" to (E) "Strongly disagree" or "This statement is NEVER true or NEVER describes me." In addition to the 150 items on the MLSP, subjects were asked to provide their date of birth; sex; anticipated college major; grades in Algebra I, Geometry, and Algebra II; and the 5-digit subject code (school, teacher, section, and participant number) which they were assigned by the investigator.

Self-reported grades in Algebra I, Geometry, and Algebra II were verified by official school records for 95% of the high school students. Such confirmation was not possible, however, for the majority of the university participants.

Analysis. The scan sheet data were transferred to tape and copied to a disk on a mainframe computer at Ohio State. Responses to the first 120 items on the MLSP were analyzed by SPSS^{X} factor analysis procedures (Nie, et al., 1983), including a varimax (orthogonal) rotation of the factor space. An examination of a scree plot of the eigenvalues indicated six primary factors, so the factor analysis was restricted to a six factor solution. Variables (items) which did not load on a factor at a magnitude of $|.35|$ or greater were excluded from subsequent analyses to facilitate item reduction and increase factor reliability. To maximize independence of the factors, items which had significant loadings on more than one factor were also excluded. The final six-factor matrix consisted of 57 variables, each of which had a factor loading of at least $|.40|$.

Reliabilities (Cronbach's α) of the six subscales were determined by item analysis procedures. Prior to these computations, responses to items with negative loadings were reversed. Subscale scores were them correlated with grades earned in Algebra I, Geometry, and Algebra II. Finally, t-tests on subscale means were performed to determine whether gender differences were present.

RESULTS

The items comprising each subscale, the number of the item on the MLSP, and the factor loadings of each are displayed in Table 1. The amount of variance in the instrument explained by the six factors totaled 40.1% (13.4%, 9.5%, 5.9% 4.1%, 3.8%, and 3.4%, respectively).

Table 2 contains the subjects' grades earned in Algebra I, Geometry, and Algebra II, separated by gender. Excluded from analysis were subjects for whom reliable information on a particular grade could not be obtained. In all three courses, females earned a greater proportion of the higher grades (A's and B's) and males a greater proportion of the lower grades (C's and below).

Table 3 contains the means and standard deviations of each subscale, separated by gender. T-tests indicated that the mean scores for males on subscales 1 and 3 were significantly higher than those for the females ($p < .001$), while the mean subscale score for the females was higher than the mean score for males on subscale 6 ($p < .001$).

TABLE 1. Items and Their Factor Loadings

FACTOR 1 (α = .87): Mathematical Sophistication

93) I'm afraid that I won't be able to do well in abstract mathematics. (.64)
79) The pace of math classes is often too fast for me. (.64)
45) My mind freezes up when I have to solve a word problem. (.64)
03) I have trouble figuring out what to do if an item on a test is worded differently than problems in the homework. (.61)
80) A course in probability and statistics would probably be very hard for me. (.59)
75) It's hard for me to see how topics in math fit together. (.58)
39) I experience a lot of anxiety when taking math tests. (.57)
106) Geometry proofs were difficult for me. (.52)
62) I would have a lot of trouble deriving a formula that I'd forgotten. (.51)
101) I feel comfortable solving problems that involve three-dimensional shapes. (-.50)
95) I want to major in something that uses as little mathematics as possible. (.47)
67) When I look at a problem, I can see it as a whole and not be distracted by details. (-.47)
06) I enjoy topics in geometry. (-.46)
42) I seem to be able to discover how things relate even if others can't. (-.45)
59) I can often look at problems and figure out the answers without using an equation or formula. (-.44)
108) I could write applications problems for an Algebra II textbook. (-.42)
107) I could probably make up some good word problems for a geometry textbook. (-.41)
92) I prefer solving problems where there is one direct, definite way of getting the answer than solving those that can be approached in various ways. (.40)
55) I learn new material best when it is presented in small pieces rather than in chunks. (.40)

FACTOR 2 (α = .80): Perceived Teaching/Learning Style Match

31) My junior and senior high school math teachers have tried to teach me in the way that I learn best. (.72)
76) My Algebra II teacher teaches/taught me in the way that I learn best. (.71)
113) Math teachers are aware of how I learn best. (.69)
111) I believe that I would learn more if the teacher would teach in a different way. (-.62)
37) I think my math teachers understand the way that I think. (.59)
56) I have a lot of respect for my math teachers. (.56)
109) I think that I've been challenged by my high school math teacher(s). (.47)

TABLE I (Continued). Items and Their Factor Loadings

FACTOR 3 (α = .76): Technical Orientation

15) I like to tinker with mechanical things. (.71)
35) I'm good at fixing mechanical or electronic items. (.69)
117) I used to enjoy taking things apart to see what made them work. (.66)
96) I believe I might like to be an engineer. (.61)
94) I'm thinking about majoring in computer science. (.61)
53) I think that I'd enjoy computer programming. (.55)
72) I enjoy science fiction. (.44)

FACTOR 4 (α = .76): Nonstandard Approach to Problem Solving

27) Classmates can't understand how I arrive at the answers to many math word problems. (.65)
26) I seem to come up with questions that other students wouldn't think to ask. (.65)
102) When I tell a math teacher how I have solved a problem, he or she will often have to stop and think about what I have done. (.61)
52) I like to try to solve problems in unique ways. (.61)
25) I look at problems differently than many people do. (.59)
70) I can often solve problems intuitively; sometimes I can't even explain how I arrived at the answer. (.55)
46) I question a lot of things that other people just seem to accept. (.49)
100) I can find easier ways to solve problems than the way the teacher has taught us. (.47)

FACTOR 5 (α = .69): Visual Preference

33) I'm an artistic person. (.71)
50) I'm considered an imaginative person. (.58)
48) I think that I'd enjoy being a cartoonist. (.54)
118) I come from a family which is talented in art. (.52)
82) I'd like to see a movie on the fourth dimension. (.47)
18) I think in pictures or images. (.47)
44) I learn more from diagrams and pictures than from words. (.44)

FACTOR 6 (α = .69): Desire for Structure

12) I put a lot of effort into my math homework. (.67)
40) When I'm solving a math problem, I like to have everything neatly organized. (.65)
71) I'm a detail person. (.49)
36) If a homework assignment seems trivial, I usually won't do it. (-.48)
04) I resent having to "show all work" on math tests and homework. (-.47)
60) I prefer specific assignments to open-ended assignments. (.45)
68) Overall, I enjoy school. (.45)
29) I write down nearly every word that my math teacher says. (.42)
38) Math classes bore me. (-.40)

TABLE 2. Subjects' Year-End Grades in Algebra I, Geometry, and Algebra II (Percent of Subjects Earning Each Grade)

	n	A	B	C	D	F
Algebra I						
Males	228	17.5	30.3	36.8	13.6	1.8
Females	259	23.9	35.9	27.1	11.2	1.9
Geometry						
Males	232	16.3	23.7	35.8	18.6	5.6
Females	249	16.9	29.3	30.5	16.5	6.8
Algebra II						
Males	230	14.8	15.7	29.5	22.2	17.8
Females	261	14.9	24.9	25.3	20.3	14.6

TABLE 3. MLSP Subscale Score Means by Gender

	Male		Female		
MLSP Subscale	*M*	*SD*	*M*	*SD*	*t*
Mathematical Sophistication	59.6	8.9	53.9	9.7	6.8***
Teaching/Learning Style Match	20.0	4.8	19.9	5.3	ns
Technical Orientation	25.3	4.7	20.0	5.2	11.7***
Nonstand. Appr. to Prob. Solv.	24.4	5.0	23.9	4.6	ns
Visual Preference	23.7	4.5	22.8	4.5	ns
Desire for Structure	27.5	5.2	30.0	5.2	5.2***

DISCUSSION

After an examination of the items comprising each of the six subscales, the following names were assigned the factors: Mathematical Sophistication, Perceived Teaching/Learning Style Match, Technical Orientation, Nonstandard Approach to Problem Solving, Visual Preference, and Desire for Structure. Reliabilities of the subscales were acceptably high, so pilot testing of a revised version of the MLSP (consisting of the 57 variables listed in Table 1) is planned. A version of the MLSP for pre-algebra students is also being considered.

Unfortunately, space does not permit a careful discussion of the gender-related differences reported above. In short, the findings suggest that females, compared to males, are less confident in their mathematical abilities (particularly in geometry); tend toward a more instrumental as opposed to conceptual understanding of mathematics; and are less attracted to careers in technology. Implications of these results and their relation to the findings of other investigators will be the subject of a future report.

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HIGH SCHOOL FEMALES' MATHEMATICS ATTITUDES - - AN INTERIM REPORT

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Educators have been increasingly interested in differences between male and female performance at all levels of school mathematics. Research has shown that attitudes are related to participation and achievement in mathematics. Attitudes of secondary school female mathematics students were compared in single- and mixed-sex environments in the initial year of a two-year study. The results presented in the interim report show more positive attitudes toward mathematics by students in the single-sex environment.

Studies seem to indicate that there are no marked differences in mathematics performance between males and females in the elementary school years. By the end of the secondary school years, however, females seem not to perform as well as males (Fauth and Jacobs, 1980; Fennema and Carpenter, 1981). There are indications that attitudes play a significant role in the prediction of females' achievement in high school mathematics courses (Aiken and Dreger, 1961).

In the last two decades educators have been increasingly interested in differences between males and females at all levels of mathematics. Sex differences have been examined by comparing achievement (Ethington and Wolfle, 1984; Ethington and Wolfle, 1986; Hanna, 1986; Rallis and Ahern, 1986). Others have compared attitudes and participation (Fennema and Sherman, 1977; Fennema, 1980; Joffe and Foxman, 1984). Research has shown that attitudes are related to participation and achievement in mathematics (Armstrong and Price, 1982; Shaughnessy, 1983) and that the traditional view of mathematics as a male domain has contributed to the decline in performance, attitude, and ultimately participation of females in high school mathematics classes (Ernest, 1976; Sells, 1980; Fennema, 1981; Fennema, 1984).

Little opportunity has existed to study changes in attitudes and achievement attributable to the inclusion of males in formerly all-female classes. The merger in 1987 of an all-female Catholic high school, D, with a similar all-male high school presented a unique opportunity to identify attitudes that prevail among females in the mathematics classroom and to chart attitudinal and achievement changes following the introduction of males.

A two-year study at D was designed to compare attitudes, participation, and performance in mathematics before and after the merger. In the first year an investigator-constructed questionnaire was administered to all mathematics students during the Spring term. The questionnaire was designed to obtain information about attitudes in the single-sex setting and also about students' expectations of the mixed-sex classroom. Additionally, performance and participation data were collected from the instructors for that term. In the second year an equivalent questionnaire will be distributed during the Spring term. At that time D will be completing its first year as a mixed-sex school.

In the initial year of study, 1986-1987, D was compared to other private mixed-sex schools to obtain further information about the effect of environment on female attitudes towards mathematics. The schools selected were H, a Catholic mixed-sex high school, and S, an independent non-denominational mixed-sex high school. Female performance in mathematics in Catholic single-sex high schools was not found to be different from that of females in Catholic mixed-sex high schools in a study conducted by Riordan (1985). Because of differences in achievement, educational expectations, and aspirations (Coleman, Hoffer, and Kilgore, 1982) between public and private schools, only private schools were chosen for this study.

All three high schools involved in the study were college preparatory. The mathematics requirement for D and H is two years, whereas for S it is three years. Female participation in mathematics by grade level in each of the three schools in the Spring of 1987 was as follows:

	GRADE			
	9	10	11	12
D	100%	90%	91%	58%
H	100	100	61	46
S	100	100	100	75

METHOD

Many instruments including the Mathematics Attitude Inventory are available for measuring attitudes toward mathematics (Aiken and Dreger, 1961; Fennema and Sherman, 1976; Michaels and Forsyth, 1977). These instruments do not provide information about female students' perception of the impact that male participation has on the mathematics classroom. In addition, none of the instruments allow for student comments, an essential component of this study. Hence, investigator-constructed questionnaires were used to measure the attitudes of females toward 1) mathematics, and 2) the inclusion of males in the classroom.

While designing the instrument several objectives were considered important. First, the instrument should determine students' like or dislike of mathematics and their self-esteem in mathematics. Second, it should ascertain their view of the importance and usefulness of mathematics. Who encouraged them to pursue courses in mathematics? Third, the inventory should determine the students' perception of who excels in mathematics; males, females, or no difference. And finally, the instrument should solicit information about their views of possible differential treatment and the impact of the participation of male students in their mathematics classroom - presently, in schools H and S, or in the future in school D.

The instrument was administered by the classroom teachers in all three schools in the Spring of 1987. As mandated by the participating schools, the alloted response time did not exceed 20 minutes. A statement prepared by the investigators assuring confidentiality was read to each class by the teachers prior to the distribution of the questionnaires. The completed questionnaires were placed in envelopes which were sealed and forwarded to the investigators.

RESULTS

The total number of responses received was 722; 353 from D, 280 from H, and 89 from S. A summary of the "yes" response percentages, comparing the single-sex, D, and the mixed-sex, M = H and S, follows. Question 5 asked the students to state their expected grade in their mathematics course. A summary of the written responses is discussed later.

	TOTAL	D	M
1. Do you like mathematics?	64.1	69.6	58.8

2. Do you think that you are good at mathematics?	56.5	63.3	50.0
3. Do you think that the mathematics you are taking is useful?	68.4	65.9	70.8
4. Do you think that it is important to take mathematics?	96.4	96.9	95.9
6. Do you plan on taking a mathematics course next year?	90.8	90.9	90.8
7. Has anyone encouraged you to take mathematics?	68.6	63.5	73.4
8. Are you comfortable asking questions in your mathematics class?	83.1	86.6	79.8
9. Do you think that (check one):			
girls are better in mathematics than boys	7.9	8.1	7.6
boys are better in mathematics than girls	8.8	9.6	8.2
there is no difference in mathematical ability between boys and girls	83.3	82.3	84.2
10. Do you think that girls and boys are treated differently in mathematics classes?	21.6	27.5	16.4
11. Do you think having boys in class:			
a) affects the atmosphere of the mathematics classroom?	54.8	74.9	35.5
b) affects how you feel about your performance in your mathematics class?	19.6	27.2	12.3
c) affects the way the mathematics class is taught?	18.3	20.7	15.9
d) affects your like or dislike of mathematics?	9.2	10.8	7.7

The analysis of the questionnaires through the application of a chi-square test produced several noteworthy results shown in Table 1. There were significant differences between single-sex (D) and mixed-sex (M) schools in the students' responses to various questions.

Females in school D appear to be apprehensive about the inclusion of males in their classes. Their responses to questions about whether having males in the classroom will affect their performance, the atmosphere, and the differential treatment all suggested apprehension. Intimidation and the need to impress were reasons given for this effect by respondents in both settings.

Respondents from D viewed males in their classroom as having a negative effect: females would feel dumb, uncomfortable, and hesitant. Respondents from M, however, expressed a more positive view: heightened enthusiasm. Although distraction was noted by respondents in both environments to have some effect on atmosphere, the females at M did so much more often than those at D.

Although males were not perceived as better in mathematics than females by all respondents (see question 9), over one-fourth of those responding said males are considered smarter. The most frequent response was that males get more attention than females.

Table 1 also shows that female students in a single-sex environment (D) have a more positive attitude about mathematics and their ability in mathematics than their counterparts in the mixed-sex schools (S and H). Most respondents stated that they felt mathematics is useful and important. The most frequently cited reasons for why it is important were 1) it will be needed in the future, and 2) it is needed every day.

Responses to question 8, referring to whether or not the respondents felt comfortable asking questions in class, indicate a difference in favor of the single-sex school. Female students in the mixed-sex schools reported having received more encouragement to take mathematics than those in the single-sex school. The respondent's family, i.e., mother, father, sibling, or other relative(s), was the prevailing source of encouragement. Parents and teachers combined was also cited, although not as frequently (51.2% family vs. 26.9% combination). Individually teachers and counselors were cited infrequently (teachers 8.8%, counselors 3.1%).

Of the respondents in both settings who said that having males in the mathematics classroom affects the way the class is taught, most said that males would be favored. The class atmosphere would be strict and the pace slower. A small percentage of respondents said that the inclusion of males would (or did) affect their like or dislike of mathematics. Some said it would be more interesting and fun, while others said they would be given less attention and feel uncomfortable.

Students were also asked what grade they expected to receive. There was a significant correlation between expected grades and whether they like mathematics, think they are good at mathematics, and feel comfortable asking questions in class, see Table 2. The responses from the respondents at D showed a higher correlation for all three variables than for the mixed-sex schools.

CONCLUSIONS

The results of the study point out differences in attitude between females in single-sex and mixed-sex mathematics classrooms. The intent of this report is not to recommend one environment over another, but rather to recommend the use of the information obtained to improve females' attitudes toward mathematics. As was shown by Fennema (1981) and Shaughnessy (1983) positive attitudes lead to greater participation. The females who participated in this study and had high self-esteem in mathematics also reported better grades.

It is important to note that respondents in both settings were quite positive about mathematics and their participation in mathematics classes. Respondents at D, however, were more positive in both respects. There could be several explanations for this. Taking the responses to all the questions in the questionnaires into account it appears that classroom dynamics contribute to these feelings.

Significant differences were found (see Table 1) between the respondents in the two settings for questions 8, 10, 11a, and 11b. These items addressed issues in the mathematics classroom. Question 8 referred to their feelings about asking questions in their mathematics class. Respondents at D indicated that they were comfortable more often than those at M. Question 10 asked respondents at D to predict whether males and females are treated differently in the classroom, and those at M to state whether there is differential treatment. The former said that there would be differential treatment more often than the latter. (Note that this prediction is not necessarily made 'blindly', but could be based on previous experiences in elementary or junior high school.)

Questions 11a and 11b are both prefaced by "Do you think having boys in class..." The first refers to their effect on the classroom atmosphere. Respondents at D strongly predicted that such an inclusion would affect the atmosphere, and their written comments focused mainly on the negative aspects. The second refers to how they will affect the females' feelings about their own performance. Again, respondents at D made more negative comments than those at M.

There were male and female teachers in all three schools involved in this study. In Riordan's investigation (1985) great emphasis was placed on the assumption that female teachers, as role models, promoted more positive attitudes in their female students. Although this may in fact be the case, all of the respondents were exposed to teachers of both sexes and a large percentage displayed positive attitudes. Additionally, only about one-third reported receiving encouragement from a teacher to take mathematics. The majority said that they received that encouragement from their family.

It is encourageing to find that, for the most part, the females studied have positive attitudes toward mathematics. This is not universal, however, and each step taken to improve the situation is important. Respondents at D were very apprehensive about the inclusion of males in their classes. It will be interesting to learn whether their fears, reservations, and other reported negative concerns actually materialize. But that must await the completion of the second year of the study.

Those students who are already in mixed-sex schools report more positive feelings about the classroom environment. Nevertheless, they too frequently mention feeling intimidated, dumb, uncomfortable, and hesitant in that environment. They also comment on the differential treatment - - that males get more attention than females. At the same time the respondents point out that the extent of the differential treatment depends on the teacher.

Many variables affect female attitudes toward mathematics and their self-esteem in that area. The more encouragement that they receive, the better. Counselors and teachers should make a special effort in this respect, for it will undoubtedly help these students to make decisions that are sometimes difficult to make because of either peer pressure or simple lack of initiative. Teachers in the classroom must be aware of the females' feelings about preferential treatment given to males. While female teachers may be very positive role models for their female students, male teachers can be that also. The goal is to encourage as many students as possible, both male and female, to participate in and enjoy mathematics.

TABLE 1
Single-sex vs. Mixed-sex

Question		df	Chi Sq	p
1	like	1	9.091	.003
2	good	1	12.607	<.001
7	encourage	1	8.346	.004
8	comfort	1	5.955	.015
10	treatment	1	12.447	<.001
11a	atmosphere	1	112.35	<.001
11b	performance	1	25.448	<.001

TABLE 2
Correlation of expected grade with question

Question	Single	Mixed	p
1 like	.47	.25	.0001
2 good	.56	.55	.0001
8 comfort	.26	.21	.0001

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CHANGING PRESERVICE TEACHERS' CONCEPTIONS OF MATHEMATICS LEARNING

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This paper presents a case study of two students enrolled in a sequence of conceptually-based mathematics courses for prospective elementary teachers. The research investigates the nature and extent of changes in their knowledge and beliefs about how mathematics is learned and what it means to know mathematics. (1)

Leaders in mathematics education - The National Council of Teachers of Mathematics, The National Science Board, The Board of Mathematical Sciences, American Association for the Advancement of Science - are calling for a reorganization of the mathematics curriculum around concept development and problem solving.

Reorganizing mathematics curriculum and instruction around concept development and problem solving poses several substantial problems. It requires that teachers have a conceptual understanding of mathematics, know why understanding concepts is important, and know how to help students gain that understanding (Resnick, 1983; Shulman, 1986; Lampert, 1986). For many elementary teachers, the limitations of their knowledge about mathematics and teaching mathematics constrains their ability to teach conceptually. These limitations originate in their own experience, as learners of mathematics and as students in elementary teacher preparation programs. If prospective elementary teachers are to overcome these limitations, they must have opportunities in their teacher preparation programs to deepen their knowledge about the nature of mathematics, children's mathematics learning (Ginsberg, 1977; Resnick & Ford, 1981; Carpenter, Moser, & Romberg, 1982; Romberg & Carpenter, 1986), and instructional practices that promote conceptual understanding.

A sequence of three mathematics courses has been developed for undergraduate education majors in Michigan State University's Academic Learning Program.(2) These courses emphasize the conceptual foundations of mathematics and actively engage prospective elementary teachers in making sense of mathematical situations. A longitudinal research project is studying both knowledge and contextual constraints in implementing a conceptual approach to mathematics in elementary classrooms. The main question informing the study is, "What is the nature and extent of changes in the types of knowledge about mathematics, mathematics learning and mathematics teaching among students as a result of these courses?"

Data for the study consist of recorded fieldnotes and video-recorded class sessions. Students who entered the Academic Learning Program in fall, 1987, periodically completed questionnaires. In addition five students were selected as an intensive sample. Data for this group include tape-recorded interviews, written assignments and exams. The first course in the mathematics sequence, devoted to an exploration of numbers and number theory and emphasizing patterns, relationships and multiple representations of problem situations, was taught spring term, 1988.

The study reported here focuses on two main questions: "What does it mean to know mathematics?" and "How is mathematics learned?" As a framework for analyzing our data we have developed three levels that correspond to different orientations to the teaching and learning of mathematics. (3) A partial description of the levels most pertinent to the discussion here is provided in Figure 1. For the purposes of this paper we have selected two students from our intensive sample to compare and contrast how their thinking is changing.

ALLISON

By the beginning of the first math course Allison had already incorporated a language of connections and representations from earlier teacher education courses. On a questionnaire item "knowing mathematics means:" she selected, "being able to make connections in the mathematical ideas that arise from different situations." But in a follow up interview the example she provided made it apparent that her interpretation of making

What does it mean to know mathematics

LEVEL 3: Using mathematical knowledge to make sense of new problem situations
:to recognize similar mathematical ideas that arise from seemingly different situations.
Engaging in mathematical inquiry
:to make conjectures, to engage in productive searches by drawing upon a repertoire of strategies and representations, to validate assertions with convincing arguments, and to communicate the outcomes of investigation to others.
Making connections among mathematical ideas
:to understand the relationship among central ideas that ground an area of mathematics.
Representing mathematical ideas in meaningful forms
:to use multiple representations to describe a mathematical situation.

Authority for knowing is internal and collective - individuals evaluate the results of exploration based on standards of the mathematics community.

LEVEL 2: Having a conceptual understanding of mathematical ideas
:to understand isolated procedures, algorithms and formulas at a conceptual level.
Describing a mathematical concept with concrete, pictorial and symbolic representations.
Solving non-routine problem situations and explaining why a particular strategy is appropriate.
Applying computational skills to situations in everyday life.

Authority for knowing is external - explanations are given but experts are the ultimate judge of correctness.

LEVEL 1: Doing rapid mathematical calculations.
Relying on memory for the correct procedure, formula or algorithm that produces the correct answer to a problem situation.
Manipulating mathematical symbols.
Meeting each problem as a new situation unrelated to previously encountered problems.

Authority for knowing is external - perceived experts are the source of knowledge.

How is mathematics learned

LEVEL 3: Students acquire the record of mathematical knowledge
:by actively participating in the construction of mathematical knowledge.
:by exploring, making conjectures, looking for relationships, reasoning and validating arguments about problem situations.

LEVEL 2: Students acquire the record of mathematical knowledge
:by listening to teacher demonstrations, manipulating concrete materials, doing guided practice exercises, and answering teacher questions.

LEVEL 1: Students acquire the record of mathematical knowledge
:by memorizing and accumulating isolated facts, rules, formulas and procedures.
:by progressing through a linear sequence of teacher-directed instruction where each idea is a prerequisite for the next, and where all ideas are treated as equally important.

Figure 1. Levels of orientation.

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connections meant something else.

For Allison connections and representations were linked to daily life applications. She referred to her experience working in a dentist office and how calculations were involved in the process of bonding teeth. She suggested using money as a way to represent decimal numbers. She selected a typical textbook word problem about a school candy sale as an appropriate problem solving situation for fourth graders. As she told us making connections means "to be able to do things with it", and that problems should be connected to "real life" things that kids are "interested in".

Allison had taken the terminology from her teacher education program and was fitting it into her framework of what it means to know mathematics. While on the surface it appeared that she was at level 3, in fact the substance suggested that she moved between levels 1 and 2 in thinking about connections and representations. This was particularly evident when she talked about learning mathematics as a "hierarchy...one idea builds from another." As an example she said, "When looking at division, working through addition and subtraction, learning rules and ideas on how that works...because when you do division you need to use subtraction."

By the end of the math course there was a significant change in the way Allison used the terms connections and representations. She talked about "seeing the relationships...one concept that really has the capacity of branching off and opening up the world of mathematics in different areas." Prime factorization was a key concept that she used to make sense of unfamiliar problem situations. When pressed about why she kept coming back to this strategy she said, "Even if it doesn't help me come up with a correct solution it gives me a way to approach the problem. It seems to appear in a number of different problems that we've done."

Multiple representations became a powerful tool for Allison. She now saw many ideas as integrated where before mathematics was compartmentalized. "Algebra I & II went together and geometry was off in its own little world...Now I can realize just how integrated it is...Like it makes other things a lot easier if you can see the geometric representation of it." As an example she talked about "seeing what a square number really looks like" and drew a dot array as a pictorial representation of the square number 9.

She assessed the appropriateness of problem situations differently. The word problem she chose earlier she now described as "simplistic...It's cut and dry. There's little thinking, just computation." What counted now were problems with a larger "purpose, and overall connections to other ideas."

At the end of the first course Allison was approaching level 3 in the substance of her thinking about connections among and multiple representations of mathematical ideas.

DENISE

At the beginning of the first math course, Denise moved between levels 1 and 2 in thinking about what it means to know mathematics. She had incorporated a language of connections. Like Allison, making connections meant how one would use the mathematics in everyday life. On the questionnaire item, "Knowing mathematics means:" she too selected "being able to make connections in mathematical ideas that arise from different situations". In the follow up interview she elaborated: "What's important is everyday uses of math in our lives. Things like how we use checkbooks, figuring percentages on sale items or for gratuities...All I can think of is when are we going to use this stuff?"

Memory plays a large role in Denise's conception of what it means to know mathematics. In several problem situations posed during interviews she was constantly searching her memory for a correct formula or procedure. With one problem she said, "It seems to me when we were in school there was a way to figure this out." With another problem, "I'm not sure if that's right but I seem to remember doing that a lot."

Denise views learning mathematics as a linear, step by step logical process. "When I think of learning math I think of

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having to learn what the symbols stand for first before you can use them and I guess the symbols seem to have a certain sequence in that you need to learn to add and subtract before you can multiply and divide." In talking about how people learn mathematics she said, "Math is hard to explore. It's logical and it's step by step."

Unlike Allison who had begun to move toward level 3 by the end of the first math course, Denise's thinking about learning mathematics changed little. She was incorporating more talk about reasoning, exploring and multiple representations. But she used these terms in a way that showed little change in her thinking. For example she said, "Reasoning is important. You need to see math as moving step by step and a person needs to be able to climb the stairs." She talked about exploring but for her that meant persisting until she got the right answer. Even in the final interview she referred to balancing checkbooks and getting change at the store as making connections in mathematical ideas.

Multiple representations meant having a repertoire of strategies to teach the same topic. "I think that one of the biggest things is having a broad range of representations to draw from...I now have many more experiences with perimeter, volume and surface area and those kinds of things from which I can take ideas...to show kids how to do it...It's like tuck this away in the back of my mind so I can remember how to use it." For her multiple representations amounted to instructional strategies and not tools for exploring the richness of a mathematical situation.

DISCUSSION

The preliminary analysis of the data from the first course suggests that these two students attach different meanings to key conceptual ideas in their teacher education program. The notions that Allison brought with her about what it means to know mathematics appear to be challenged by this course and she is moving toward a level 3 orientation. Denise, on the other hand, seems to be adapting these conceptual ideas to fit the framework of beliefs she brought to the course.

The case study of these two students demonstrates the complexity of influencing prospective teachers' beliefs about what it means to know mathematics and how mathematics is learned. We will continue to follow our intensive sample of students over the next two years as they complete the mathematics sequence, student teaching and their first year in the classroom to chart and account for changes in their beliefs and practice.

Endnotes

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(2) Perry Lanier is director of the project. Glenda Lappan is associate director and principal designer and instructor for the sequence of mathematics courses.

(3) This framework grows out of theoretical work of our colleagues Deborah Ball and Anne Madsen-Nason. We gratefully acknowledge their contribution.

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AN ANALYSIS OF IN-SERVICE TEACHERS' MATHEMATICAL BELIEFS: A COGNITIVE DEVELOPMENT PERSPECTIVE

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The purpose of this research is to examine the mathematical beliefs held by three high school mathematics teachers who are at different levels of cognitive development based on Perry's (1970; 1981) theory. The theory predicts that each of the three teachers use different thought structures in making meaning out of mathematics. In this research we compare and contrast between developmental levels these teachers' beliefs about mathematics. Two themes central to Perry's theory - the nature of truth in mathematics and the role of authority (teacher, professor, etc.) in understanding mathematics - are used as the basis for interviews that explore the mathematical beliefs of the teachers.

Methodology

Teachers participating in this study were all graduate students in a MAT program in the Department of Mathematics and Statistics. Each completed an open-ended instrument designed to assess their level of cognitive development. They were selected for this study based on this assessment because they represented different positions on the Perry Scheme and they had all been teaching less than five years. No effort was made to control for sex.

Two words of caution are necessary. First, readers unfamiliar with Perry's theory might consult descriptions of his work, as familiarity with the theory is assumed. Second, comments about an interviewee's developmental level should not be interpreted as a label for that person, but rather as a shorthand method to indicate there was evidence of use of thinking structures consistent with that developmental level.

Interviews

1. Adam -- In his essay Adam used thinking structures consistent with the dualistic thinker, indicating he would think about mathematics in terms of right-wrong, black-white dichotomies where there is either a right or a wrong answer to any situation. The dualist would also expect authority (teacher, professor) to know these right answers (Stonewater, Stonewater, and Perry, 1988).

In our interview with Adam it is important to note what was not said, as much as what was said. His answers are usually short, without embellishment, and do not express a very complex view of mathematics or learning mathematics. He is very direct in telling us he believes there is one right answer in mathematics:

> I think there's one right answer, like operations in Algebra I with two plus three times five. If you add two plus three you get one answer times five. If you multiply first, you get a different answer. So we have to have some rules in mathematics.

He further indicated that all mathematics problems, not only Algebra I problems, have one right answer - "As far as the overall answer, the final answer, yes...."

In a further probe, the interviewers attempted to set up a situation in which the potential for multiple truths might exist: "You have two professors, both of whom you admire, both competent in your judgment. And they disagree about a problem. Where's the problem?" Adam's response to the question illustrates the interdependence of his beliefs about truth in mathematics and his beliefs about professors as authority in knowing truth in mathematics. He is stumped by the questions and cannot respond:

> I don't have an answer for that. As far as my point of view, as far as being a masters graduate student right now, I respect my professors, I always have, I shouldn'tquestion their ability. In undergraduate school we accepted what he said as Gospel,the truth, and that situation would be very difficult.

Adam's beliefs are evidently in conflict. There is one right answer, authority always knows the truth, but authorities disagree. If authority knows "Gospel, the truth", then Adam is confronting a situation he cannot resolve.

To summarize, Adam believes there is one right answer to mathematics problems and believes that professors are the ultimate knowers of truth in mathematics. He is unable to handle situations which cause conflict between these beliefs.

2. Beth -- Beth's essay was rated early multiplicity, indicating more complex thinking structures than those Adam used. The early multiplist recognizes multiple perspectives or the possibility of more than one right answer, but sees this multiplicity as temporary and confusing. Resolution of the confusion is sought from authority, who still knows the one corract answer. Multiplists sometimes describe simple processes as the way to move through the diversity of multiple rights to the one right answer. In Beth's interview, we see evidence of early multiplistic reasoning structures, but unlike her essay, there is also evidence of late multiplistic reasoning. Thinking structures evident in late multiplicity are more complex and are usually expressed one of two ways: certain things have right answers while others are decided by opinion or interpretation alone, or via a second thinking structure, certain things have right answers, while others have to be decided by logical reason. In this second structure, students have learned to model from authority the kind of

thinking that is truely contextual and the hallmark of mature mathematical thought, yet still incomplete and lacking. In both of these late multiplistic structures, authority is still important, but much less so than in previous stages.

Beth's thinking is more complex than Adam's. Early multiplistic structures are seen primarily in her heavy reliance on authority for the right answer in her confusion over teaching decisions.

> I mean this is it, it is isolating, there is nobody in here to tell me what to do day to day.... Last year I was continually asking advice from those who I respect around here and the administration and the answer was always use your own best judgment. And I didn't feel that I had good judgment. I didn't feel that I had judgment period. I
>
> tried to do what I thought was best. And then again I don't know how I'm going to tell. At this point I do not know, I have no idea, who might know.

But Beth also shows evidence of using late multiplistic structures in describing "truth". Here we see right answer vs. a logical reasoning approach:

> I think that in calculation problems there is one right answer and no matter how you go about trying to justify a wrong answer there is only one right answer. But ther are other situations where, depending on how you set the problem up, or how you interpret the situation...you can set that model up to come out with a particular conclusion. And you may have a valid hypothesis and your model may be correct and your mathematics may be correct and you may come up with an answer that may be different from someone else's correct hypothesis and correct interpretation andthey may be different and I don't see a problem there.

When asked how she would judge between multiple answers, she faced a dilemma. In the following quote, she starts with a late multiplicity structure ("own opinion"), but ends with "one right answer" early multiplicity structure.

> I would go with my own opinion, I would go with the way that I saw it, but I would walk away saying that I don't think that that is right... because my interpretation is just my interpretation and ... if I talked to that person first maybe they could of convinced me that the other one was right or maybe I would of had an opinion or maybe I wouldn't of.

Over and over again in Beth's interview, she stresses reliance on external authority to help her decide what is "right":

> . . . and if they both made really good sense I would look to the instructor to see what they were going to do and if they didn't do anything either then I thought they both must be reasonable . . .

In summary, Beth's beliefs about mathematics are more complex than Adam's. She describes "knowing" in mathematics in three different ways: for some problems there are right answers, for some a system is used (hypothesis based on a model), and for others interpretation is used. In some situations she seems to believe that authority knows what is best to do, while in others she relys on her interpretation. Throughout all of the interview, however, there is a great deal of uncertainty and confusion.

3. Carl -- In Carl's essay relativistic thinking structures were dominant. His interview was consistent with the essay. He describes what he believes about mathematics:

> I think it gives us reason for accepting things. I think it gives us a verification of accepting things.... Mathematics is made up of axioms or things you got to believe, you have got to believe those first. I guess you have got to convey that to them (the students), that's what mathematics is. It is not that we go out and do things with numbers or something but there is a system involved. So I think proof is a way of verifying the system and going back to what you believe.

Carl sees proof as the process for verification - not authority, not interpretation, not opinion. He connects basic assumptions and the methodology of proof in a unified whole - the system takes you back to what you believe.

When discussing the importance of proof in the mathematics curriculum, Carl provides a context and reasons for his answer. Note his memory of authority as having the right answer vs. his personal understanding now of why he needs to understand proof:

> Probably the same but for different reasons. Back in undergraduate (school) I knew that the professor was giving me these proofs for a reason, but didn't really know what the reason was, but I knew it must be something good. Where now I understand why I need that. But also I understand that we need to make this more applicable (for our students), so I'm now still in the middle but for different reasons. I think we need to apply mathematics but at the same time we need to apply concepts ... to keep the construction together.

Except for the above reference to his undergraduate professor, Carl never directly discusses authority. He does refer to a former professor who "opened up new ways for me to think and to learn" and refers to himself as a "facilitator" when describing himself as a teacher. Thus, while not a direct statement about how he views other authority, he describes himself in the role of authority:

> I guess to be there to pose problems, to get them thinking, ... to keep the structure there, to help them out when they have questions. But not as the only source. And not there to limit them to what they can do. I'd just like to encourage them, to open them up ....

In summary, Carl believes that mathematics is a system based on some fundamental accepted concepts and that proof is the glue that keeps the construction together. He also believes that the system can be used for verification of what we believe and that we should be concerned with both the theory and the application parts of mathematics. Indirectly, it seems that he views authority as a helper with the process of learning mathematics.

Summary

In these interviews we see three teachers, each of whom uses different Perry theory thinking structures. While interviews with three teachers are insufficient evidence from which to draw conclusions about the relationship between cognitive development theory and mathematical beliefs, we do feel the following tentative observations are in order. These observations will form the basis for an extended research study.

1. There were fundamental differences in the three teachers' expressed beliefs about the nature of truth in mathematics and the role of authority in understanding mathematics. This suggests, but only preliminarily as there are rival hypotheses, that cognitive developmental theory might provide at least a partial theoretical basis for the study of beliefs in that it offers a way of understanding belief differences.

2. There was a surprising internal consistency for each interview in the content of the beliefs expressed with respect to truth and authority. In other words, none of the teachers expressed beliefs that were contradictory to the thought structures for their developmental level. We stop short of suggesting that this observation provides evidence that their beliefs are either a function of developmental level or that their beliefs can be predicted by their developmental level. At the same time, the internal consistency suggests that further study examining this issue is warranted.

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First and Second Graders' Ability to Use Schematic Drawings in Solving Twelve Kinds of Addition and Subtraction Word Problems

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First and second graders learned to fill numbers from all kinds of addition and subtraction problems into schematic drawings of four different major problem types (Change-Get-More, Change-Get-Less, Put Together, Compare/Equalize). On problems with sums ≤10, first graders improved significantly on every problem type on both strategies and answers, and second graders did the same for problems with sums between 10 and 18. Average- and high-ability second graders and high-ability first graders virtually reached ceiling on all problem subtypes for problems with sums between 10 and 18, and low-ability second graders and average-ability first graders did well on many such problem subtypes. High-ability second graders solved most subtypes of problems with 3-digit numbers requiring one trade. Thus, children in the United States are capable of solving addition and subtraction word problems that are much more difficult than those given in standard US texts.

The kinds of addition and subtraction word problems given in first and second grade textbooks in the United States are restricted to very simple types that many children can solve as early as kindergarten (Stigler, Fuson, Ham, & Kim, 1986). This is in contrast to first and second graders in the Soviet Union, whose textbooks contain a fairly equal distribution of word problems across four major problem types each divided into three different problem subtypes differentiated by the unknown number (Stigler et al., 1986). DeCorte, Verschaffel, and Zwerts (Zwerts cited in DeCorte & Verschaffel, 1985) and Willis and Fuson (1986) reported in individual classrooms success in teaching first and second graders to use schematic drawings that differed for these different main types of problems. The present study expanded the approach used in Willis and Fuson to cover a wide range of different sized numbers and mathematical ability. The whole range of 12 subtypes contained in the Soviet texts was included and systematically tested. This exploratory study examined whether classroom teachers could use the method and whether the whole range of problems included in the Soviet texts were within the zone of proximal development of US first and second graders.

METHOD

All of the first graders ($\underline{n}$ = 122) and second graders ($\underline{n}$ = 102) in two schools in a small city on the northern border of Chicago participated in a year-long instructional study focused on learning to solve addition and subtraction word problems. Children in these schools are grouped in mathematics classes according to mathematics ability. There were 6 classes of first graders, one class in each school of low, average, and high ability. In one school there was one class each of low, average, and high ability second graders; the other school had only 2 classes of second graders, one of low and average and one of average and high ability.

Instruction consisted of three separate phases, addressed to problems with sums ≤10, sums between 11 and 18, and 3-digit sums involving a trade (carry, regrouping). Schematic drawings for Change, Put Together, and Compare/Equalize problems were used in each phase (see Figure 1). In the first phase teachers read word problems aloud, and children used chips to model the problem on sheets containing large schematic drawings. In the second phase, teachers read problems aloud as necessary, and children filled the numbers of the problem into small schematic drawings (see Figure 1) and then used the drawing to decide whether to add or subtract to solve the problem. In the third phase, teachers read problems aloud as necessary. Children filled the 3-digit numbers into small schematic drawings, used the drawing to decide whether to add or subtract, wrote the addition or subtraction problem in column form, and solved the problem. Within each phase teachers moved through 3 levels of problem difficulty: simple addition problems (Change-Get-More: missing End and Put Together: missing All), simple subtraction problems (Change-Get-Less: missing End, Compare/Equalize: missing Difference and Put Together: missing Part), and difficult addition and subtraction problems (the remaining six problem subtypes).

Children were able to solve symbolic addition and subtraction problems considerably above grade level because they received during the year special instruction in adding and subtracting single-digit numbers with sums to 18 and multidigit numbers with trading (Fuson & Briars, 1988; Fuson & Secada, 1986; Fuson & Willis, in press). The word problem instruction was new to all teachers in the study. Three of the teachers also had to learn the symbolic addition and subtraction instruction, a considerable burden of new learning. Children's work with word problems was to be integrated with their work on symbolic addition and subtraction problems (e.g., 14 - 8 = ?). The push sheets of schematic drawings were used with sums <10 to give meaning to the + and - symbols. All children began with this first phase, and the classes moved through the instruction as the teacher judged the ability of the children would permit.

At the beginning of school all children were given over four days a 24 problem pretest containing two problems of each of the 12 problem subtypes. The first-grade pretest contained sums ≤10, and the second-grade pretest contained sums between 10 and 18. The same tests were given at the end of the year as posttests. The first graders also received a 24-item posttest with sums between 10 and 18, and the second graders received a similar posttest with 3-digit sums requiring one trade (regrouping). Within each test the six Compare problems were divided between 3 ordinary Compare problems (one with each missing position) and 3 Compare problems with questions in the Equalize form ("How many more does Mary have to get to have the same number as Joe?" and the relevant variations for the other two missing positions).

RESULTS

Low-ability first graders only began the simple addition problems with sums between 10 and 18. The teachers of these classes happened to be the weakest of the first grade teachers and had low expectations of these children; stronger teachers might have successfully gone farther (in previous years stronger teachers had successfully completed symbolic addition and subtraction with sums to 18 with these children, see Fuson & Secada, 1986). Average-ability first graders completed the simple subtraction problems with sums between 10 and 18. Both of these teachers had asked to teach multidigit addition with trading to their children (results are reported in Fuson & Briars, 1988). This took a considerable amount of time, so one might ordinarily be able to work with all twelve problem subtypes with sums ≤18 High-ability first graders completed work with all 12 problem subtypes for sums between 10 and 18. The high and high/average second-grade classes completed all 12 problem subtypes with 3-digit numbers. The average second-grade class just began the difficult problems with 3-digit numbers, the average/low second-grade class just began the simple subtraction problems with 3-digit numbers, and the low-ability second-grade class only completed the difficult subtypes with sums between 10 and 18 because this class did not begin the subtraction multidigit work.

Pretest-posttest differences were tested for each main problem type separately for both strategy scores (choosing the correct solution procedure) and answer scores (getting the correct answer). Significant *t*-tests indicated that on problems with sums ≤10 first graders in every class improved significantly on every problem type on both strategies and answers and that on problems with sums between 11 and 18 second graders in every class improved significantly on every problem type on both strategies and answers (see Table 1 for posttest means). Analyses for each missing position for each problem type indicated considerable differences in improvement depending upon grade and ability level. The low-ability first-grade children showed significant gains in the problem subtypes shown in previous research to be easiest (the simple addition problems: Change-Get-More: missing End and Put Together: missing All). The average- and high-ability first-grade children gained significantly in the more difficult missing positions (Change-Get-More and Change-Get-Less: missing Start and missing Change, Put Together: missing Part, Compare/Equalize problems). The second graders showed significant increases in all but the simplest missing positions, where they were already high on the pretest. For sums between 10 and 18, most second-grade classes reaching ceiling on most problem subtypes (see Table 1).

On the posttest all children showed an effect of the size of numbers used in the word problem, but this effect differed by grade and, for the first graders, by ability level. Both grades showed overall fairly good competence with the larger numbers given at their grade level (see Table 1). For every class of second graders, both strategy and answer scores were better for problems with sums between 11 and 18 than for the 3-digit sums. This strategy difference was quite low in the higher ability classes and was considerably larger for the lower ability classes. Both classes of first graders did considerably better on the sums between 11 and 18 than on those with sums ≤10, reaching ceiling on most problem subtypes for the former. The first-grade classes of average ability showed little effect of size of numbers.

CONCLUSION

The combination of instruction on addition and subtraction word and symbolic problems indicates that children in the US can learn very much more about addition and subtraction than is commonly contained in US textbooks (cf. Fuson, Stigler, & Bartsch, in press). First graders can learn to add and subtract single-digit numbers with sums ≤18, and second graders can learn to add and subtract 3-digit numbers requiring trading (most children actually generalized such addition and subtraction to much larger numbers, see Fuson & Briars, 1988). With instruction and practice children in the US can learn to solve a much richer range of addition and subtraction word problems than are ordinarily given to them. For sums <18, high-ability first graders and most second graders can learn to solve the whole range of problems identified by American researchers (e.g., Carpenter & Moser, 1983; Riley, Greeno, & Heller, 1983) and used in Soviet textbooks (Stigler et al., 1986) and children of lower ability levels can make considerable progress with almost all problem subtypes. The use of schematic drawings clearly is a powerful approach to word problems. Whether other methods are even more powerful must await comparative studies.

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Table 1

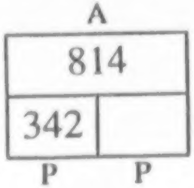
Posttest Means and Ranges by Size of Number and Class

	Small Number				Large Number			
	Strategy		Answer		Strategy		Answer	
	Mn	Range	Mn	Range	Mn	Range	Mn	Range
First Grade Mean	74	65-81	70	62-78	83	79-87	78	74-83
High-Ability	78	69-85	75	69-81	92	87-96	90	84-93
High-Ability	74	55-84	72	55-80	95	93-97	90	88-93
Average	72	67-77	65	62-70	71	65-77	65	62-71
Average	70	67-79	67	61-79	74	69-77	68	63-73
Second Grade Mean	96	93-98	94	90-97	79	75-84	63	63-69
High Ability	98	96-100	97	95-98	92	88-98	84	98-93
High/Average	98	97-100	98	94-99	89	88-89	79	76-84
Average	94	91- 95	93	91-95	73	67-81	50	42-58
Average/Low	92	87- 97	86	79-95	63	58-66	37	34-41

Note. The mean is the mean across all 24 problems, and the range is across the means for the 4 main problem types (see Figure 1). Scores are percentage correct strategy (added or subtracted as needed) and percentage correct answer (obtained correct answer). Small number for first graders means sums ≤10 and for second graders means sums between 11 and 18. Large number for first graders means sums between 11 and 18 and for second graders means 3-digit sums requiring one trade (regrouping). The low-ability classes are not included here because instruction for large numbers was not begun.

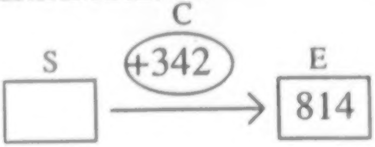
PUT-TOGETHER (COMBINE): missing SECOND PART

(All) (Part) (Part)
Jon and Bill have 814 toys altogether. Jon has 342 toys. How many toys does Bill have?



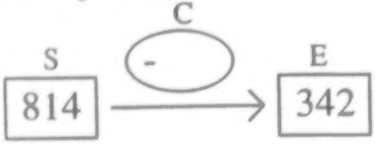
CHANGE-GET-MORE: missing START

(Start) (Change) (End)
Jon had some toys. Then Bill gave him 342 more toys. Now Jon has 814 toys. How many toys did Jon have to start with?



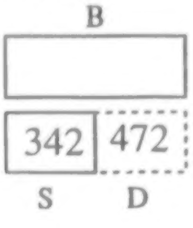
CHANGE-GET-LESS: missing CHANGE

(Start) (Change) (End)
Jon had 814 toys. Then he gave some toys to Bill. Now Jon has 342 toys. How many toys did Jon give to Bill?



COMPARE: missing BIG

(Small) (Difference)
Jon has 342 toys. Bill has 472 more toys than Jon has.
How many toys does Bill have?
(Big)



A First Look at the Long Term Effects of Conceptually-Based Instruction on Decimals

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The paper examines the long term effects of a two-week instructional unit on the meaning of decimal fractions. A classroom of fourth graders participated in the special instruction that moved from introducing decimal notation as a symbolic record of concretely-represented quantity to adding and subtracting decimals. Written tests and interviews given a year later suggested that (1) some students retained a meaningful approach to solve decimal problems and (2) the experimental subjects, as a group, solved more problems correctly and used a meaningful approach more often than a matched group of conventionally taught peers.

There is general agreement among those investigating students' mathematics learning that most students develop an overly mechanical approach to mathematics. There also is general agreement that more conceptually-based instructional approaches are needed to increase students' understanding of the subject. Although there are many theoretical advantages for instruction that emphasizes conceptual knowledge, there is little empirical data on the precise effects of such instruction. This is particularly true for long term effects, even though it is the long term effects of specific instructional approaches that are of ultimate interest.

In this paper we report some initial data on the year-long effects of a two week conceptually-oriented instructional unit on decimal fractions. We are interested in two questions: (1) Do short-term changes in the processes students use to solve problems remain stable over time, even when the students receive intervening instruction of a more conventional kind? and (2) Do students who have developed conceptually-based processes to solve problems exhibit a higher level of quantitative reasoning and performance than their conventionally taught peers?

Method

Sample

One classroom of 19 fourth graders received the special instruction on decimals. The students, representing a variety of ethnic groups, had percentile ranks on the mathematics portion of a standardized test ranging from 21 to 95 with a median of 65. The teacher described the class as average to below average in mathematics ability. Six students, chosen by the teacher as representative of the class, composed the interview group.

One year later 12 of the 19 students remained in the school and were available for testing. Five of the six interview subjects were available for interviewing. A group of 12 fifth graders who had not received the special instruction in fourth grade but who were in the same fifth-grade classrooms as the experimental subjects, were selected as control subjects by matching their mathematics percentile ranks on the standardized test. A subset of five students were selected as interviewees by matching their mathematics percentile ranks with those of the five experimental group interviewees.

Instruction

The special instruction in grade 4 consisted of 10 consecutive lessons that introduced students to the meaning of decimal fraction symbols through hundredths. The sequence of activities moved from connecting written decimal symbols with quantity, represented concretely with Dienes base-10 blocks, to developing procedures to add and subtract decimal symbols by paralleling actions on the blocks. A theoretical rationale for the instructional approach is contained in Hiebert (in press; see also Kaput, 1987); a description of the instructional activities can be found in Wearne and Hiebert (1988, in press).

Experimental subjects received no further instruction in decimal numbers in grade 4. In grade 5, the teachers reported that both the experimental and control subjects received instruction on decimals based entirely on textbook material. Some students in each group used a fourth-grade text; the rest of the students used the first decimal chapter of a fifth-grade text. Both texts contain the usual material on ordering decimals, rounding decimals, and adding and subtracting decimals. All types of exercises included items with tenths, hundredths, and thousandths.

Assessment

All students in the experimental group were given nearly parallel forms of written tests five times: before instruction, after the first five lessons, after nine lessons, six weeks after the instruction, and one year after the instruction. The interviewed subjects received individual interviews at the same times as the written tests. The control students took the written tests and were interviewed individually at the same time as the fifth tests and interviews were given to the experimental students.

Two kinds of assessment items were used on the written tests and in the interviews: (1) direct measures were tasks like those that were discussed during the special instruction, and (2) transfer measures assessed the use of the instructed quantitative reasoning processes in a novel situation. The use of the labels "direct" and "transfer" for the tasks on the final assessment a year later may be misleading because textbook instruction in fifth grade covered some of the transfer tasks (eg., ordering decimals). However, we still use the term transfer for these tasks because we are interested in the transfer of the processes used to solve the tasks, and the processes promoted during special instruction are different than those exemplified in the textbook exercises.

Students' explanations during the interviews for why they did what they did provided the data for examining the kinds of solution processes students used. Explanations for responses were classified into one of two categories: (1) explanations that indicated reasoning about the quantities represented by the symbols, and (2) explanations that indicated the recall and use of a rule that was not understood or the treatment of symbols based on surface features. Explanations for all responses were coded by one rater. Explanations of one experimental subject and one control subject, selected at random, were coded by a second rater. Interrater agreement was 95%.

Results

Short-Term Effects

After the special instruction, 79% of the responses were correct on the direct measure written test items and 45% of the responses were correct on the transfer written test items. All of the six interview students explained their responses to the direct measure interview items by referring to the quantitative values of the symbols, and an average of four students used these quantitative reasoning processes on the transfer tasks. The brief summary of short term effects are presented here only to provide a baseline context for interpreting the long term effects. A fuller description of the fourth-grade data is contained in Wearne (1988).

Long-Term Effects

The first question is whether the cognitive processes promoted during the conceptually-based instruction were retained and used over time? Profiles for each of the five subjects remaining in the sample a year later are shown in Figure 1. All students acquired and used conceptual, quantitative reasoning strategies to solve the three direct measure tasks given on all five interviews (after they became direct measures, i.e., after they had been discussed during instruction). Two of the five students continued to use quantitative reasoning to solve all three tasks a year later.

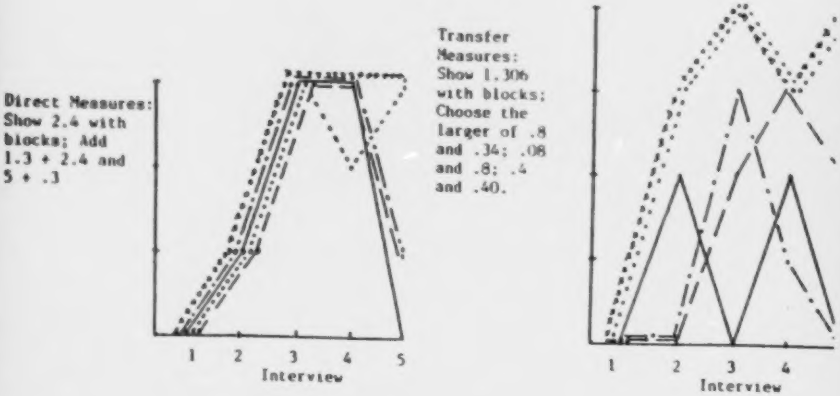

Figure 1. For each subject, number of responses explained by quantitative reasoning

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Greater individual variation is evident on the transfer tasks. The two students who apparently had acquired the conceptual processes in a more enduring way (indicated on the direct measures) showed a similar high profile on the transfer tasks and even extended slightly the use of the conceptual processes from the six week delay interview to the year delay interview.

The second question is in what ways did students who had acquired the conceptual processes during the special instruction a year earlier differ from their conventionally taught peers? Table 1 compares the written test performance of the experimental and control groups on four direct measures (write the number shown with blocks [2.13]; 5.1 + .46; 4 + .3; .5 + .21) and seven transfer measures (write the number shown with blocks [3.205]; circle the larger of .42 and .5, 4.528 and 4.53, .7 and .07, .6 and .60; write a decimal number between .4 and .5; .38 = .3 + _____). No tasks like those included as transfer tasks had been presented or discussed during the special instruction. There are a total of 48 direct measure responses and 84 transfer measure responses. Statistical tests were not run on the differences between the groups because of the small sample sizes.

Table 1

Number of Correct Responses on Written Test Items

	Experimental Group (n=12)	Control Group (n=12)
Direct Measures (4 Items)	31	7
Transfer Measures (7 Items)	21	14

Table 2 shows the performance and explanations of the interview subjects on a parallel set of tasks given during the interviews. As a group, the experimental subjects used the conceptual processes to give explanations based on appropriate quantitative values of the symbols about half the time whereas the control subjects gave equally meaningful explanations about one-eighth of the time.

Table 2

Number of Correct Responses and Explanations on the Interview Tasks

	Experimental Group (n=5)	Control Group (n=5)
Direct Measures (4 Items)		
Correct Responses	13	3
Quantitative Reasoning Explanations	10	0
Transfer Measures (7 Items)		
Correct Responses	12	2
Quantitative Reasoning Explanations	16	7

Discussion

Given the small sample size and the limited range of tasks, the summary observations should be interpreted as plausible hypotheses rather than conclusions. It appears that conceptually-based instruction, even just a two-week unit, assists students in constructing meaning for written symbols and in using meaning to solve tasks. Further, this approach is still useful for some students a year later, even though intervening instruction does not emphasize meaning. Not surprisingly, it appears that students who develop meaning more solidly at the beginning are more likely to continue using solution processes that depend on these meanings. The superior performance of the experimental group could be explained, of course, by the fact that in fourth grade they received more instruction on decimals (of whatever kind) than the control group. Of greater interest, however, is the more frequent use of quantitative reasoning by the experimental group, even on the transfer tasks. These data are best explained by intervention of a particular kind.

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Promoting Good Mathematical Thinking Through Writing in the Mathematics Classroom

by

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ABSTRACT

Recently a great deal of attention has been focused on using writing in the content areas. While mathematics has been slow to see the value of such seemingly unrelated activity, it is becoming increasingly apparent that writing exercises can enrich the mathematics learning environment. In this paper we discuss how the inclusion of a writing component can further the goals of a constructivist approach to teaching mathematics. Specifically, we outline briefly the process and goals of a constructivist mathematics class and the role that writing can play in such a class. This is followed by a description of SummerMath, where the writing project under discussion was carried out. We then describe in detail the design and implementation of the pilot writing program. Finally, we assess results and future directions.

Writing in the Constructivist Mathematics Class

Constructivist mathematics begins with the belief that people construct an understanding of mathematical concepts through their own experience (von Glasersfeld, 1983). Teaching approaches based upon this belief emphasize the activity and involvement of the student and focus on understanding concepts rather than rote learning. The role of the teacher moves away from that of knowledge-giver, as in the traditional classroom, to that of guide or mentor. Problem solving experiences are set up to provide students with the opportunity to explore and construct the concepts to be studied. Students are engaged in discussion of the problems presented--in small groups, in large groups, and in dialogue with the teacher--helping them to clarify and expand their understanding of the concepts involved. They thus learn not only specific mathematical material, but develop effective ways of approaching new material, making them more independent and confident problem solvers, able to approach future learning situations more productively (Simon and Schifter, 1988).

In order to engage a student in the process described above, the teacher must first find where the student is mathematically grounded. That place becomes the starting point from which to explore new mathematical territory. For some students, by the time they reach high school their long-term experiences of failure, pain, and fear have created barriers that preclude access to any mathematical groundedness. In such cases, major shift in students' relationship to mathematics is prerequisite to substantive learning of mathematical content.

Turning to a more familiar and often more comfortable mode, such as writing, can provide a sense of security to a math-anxious student. Alternating among various modes of discourse (writing, talking, drawing, and symbolic representation) builds bridges between formal mathematics and knowledge embedded in students' everyday experiences; insistence that mathematically valid thought is restricted to rule-governe manipulation of strings of symbols keeps the mathematics insulated from their person knowledge.

Writing across the curriculum is increasingly being seen as a way to foster student development as well as to promote learning in the various content areas. Peter Elbow's (1981) pioneering work, *Writing with Power*, has made it clear that writing helps students gain access to their ideas and promotes critical thinking. Furthermore, writing helps students see connections between affective and cognitive domains so they can begin to understand the ways in which their feelings hinder or facilitate intellectu processes.

Recent work with writing in mathematics classes has shown that students gain understanding of mathematical ideas and an appreciation of the complex nature of mathematical material (Nahrgang and Petersen, 1986; Watson, 1980). The goal of th project described in this paper was to help students develop a more personal connectio to mathematics, thus making mathematics more accessible.

THE SUMMERMATH PROGRAM

SummerMath is a six-week residential program in mathematics and computin held on the Mount Holyoke College campus for high school age women. The program is neither remedial nor accelerated, but aims at a deep understanding of basic mathemati

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concepts for students at all levels. The core of the program rests on a curriculum designed to foster the construction of mathematical understandings on the part of the student. Through group problem solving and verbalization of the process, students enhance their own and each other's understanding and achieve greater flexibility and persistence in problem solving.

Students attend 4.5 hours of math and computer classes every day, including 1.5 hours in a class called Fundamental Mathematical Concepts (FMC).

THE SUMMERMATH PILOT WRITING PROJECT

Background

The writing project, implemented in SummerMath 1987, was designed cooperatively by three staff members, each with her own area of interest and expertise. The Director had taught for several years in a Language and Thinking Workshop at a liberal arts college and was interested in extending the applications of writing into the mathematics domain. The Coordinator of FMC had had extensive experience teaching remedial mathematics and was scheduled to teach a section of the least advanced students. She was interested in ways that writing could serve students who had had negative experiences with mathematics. Finally, the assistance of a writing teacher, already associated with the SummerMath program in another capacity, was enlisted to design specific writing exercises.

The students whose class was to include a writing component had already indicated that they had had years of bad experiences with mathematics. The goal of the project was to use creative writing to help students alter their relationships to mathematics so as to approach mathematics from a more empowered position.

Procedure

In one section of the most basic level FMC classes--consisting of 9 students, one teacher, and one teaching assistant--half of a mathematics class each week was devoted to writing. There was a total of five 45-minute sessions devoted to writing during the course of the SummerMath program.

The writing exercises were led by the writing teacher who attended class for the entire class period on the day on which the writing was to occur. The regular teacher and assistant remained in the classroom and participated in the writing assignment. During the 45-minute session, an initial exercise was given with ample time allowed to write a response. Following the writing task, the remainder of the period was devoted to sharing and discussion of the writings.

Following is a list of the five writing exercises given to the students:

1. (Two characters, one male and one female, both mathematicians, were created by the class.) Select one character and write in the character's first person about a work day.
2. a. Write a true story about a situation in the past involving a math teacher that made you angry. b. Rewrite the story as if you were the teacher and change it so you like the outcome.
3. Imagine and write about a world without mathematics.
4. If you were a mathematics teacher, how would you teach.
5. Guided imagery exercise: You are back in your regular school mathematics class. Write a story about it.

Results and Discussion

In general, by the end of the SummerMath Program each year, mathematics instructors see evidence of change in students' attitudes toward math. Students who at the beginning of the summer avoided and were intimidated by mathematics, are now willing to take on challenging problems and want to discuss their ideas about them. Frequently they are reluctant to put their work away at the end of the period because they want to pursue their own train of thought.

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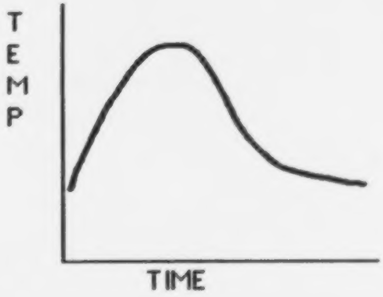
A particular type of change took place in the class engaged in the writing project--one that involved bringing personal experience of a kind not usually admitted into a math class to bear on mathematics problems. The following anecdote illustrates this change:

During the fifth week of the program, the teacher began a unit on functions and graphing. The lesson was introduced with a graph of "TEMP vs. TIME:"



Students were asked to describe a situation for which this graph could be a representation. The instructor, who had used this lesson a number of times in the past, usually heard such suggestions from students as, "The temperature during a day rises and falls," or "A cup of coffee is heated and then falls back to room temperature." However, this class chose "TEMP" to stand for temper, rather than temperature. The first suggestion was:

> I see the mood of a SummerMath student through the duration of the program. When she arrives, everything is fine. But then she has to adjust to all the problems of living in the dorm, and all the frustration of being asked too many questions in math class. Then, after about the middle of the program, she begins to realize how much she's learning and gets used to the program. In the end she's feeling calm, but is sad to be leaving her friends.

This and other, similar, examples suggest that students in the writing program experienced a significant shift in their relationship to mathematics. Rather than looking at a mathematical representation with paralyzing anxiety, these students could now link mathematical abstraction to knowledge embedded in intimate aspects of their lives.

Nine months after the end of the program, follow-up questionnaires solicited from the students in the pilot project revealed the impact of SummerMath on their math studies at their home schools. Reports indicated improvement in their grades from C's, D's, and failing to A's and B's. They also reported increased confidence, not only in mathematics, but generally:

> The confidence which helps me now achieve better in math has also helped me in other aspects of my life....I never realized before how much my problems with math burdened me psychologically.

Certainly, this pilot project was not a controlled experiment. The creative writing comprised about 10% of the time in a constructivist math class--for these students, a first experience of a math class that emphasized their own thinking. Students' comments responded to that aspect of the class as well.

> The philosopy of enforcing a student's confidence through having them help themselves find answers was a valuable lesson. I remember drawing graphs and diagrams to help me...see the workings of math....I remember what I was taught and I integrate this into my math career.

Responding specifically to the writing assignments, one student said:

> Creative writing class helped me to deal with a lot of negative feelings I had concerning math, a subject which always marred my report card. [It] let me reveal my true feelings about math. Because I enjoy writing, I felt comfortable with myself and math.

CONCLUSION

Informal analysis of this pilot project suggests that a portion of the success achieved by some FMC students can be attributed to the writing exercises and subsequent discussions. Students' attitudes toward themselves and mathematics improved, and they now felt more comfortable and competent in their math classes.

Of course it is also important to examine the context of the creative writing lessons--a constructivist math class. Along with students' examination of their relationships to mathematics, they were constantly challenged to think about, talk about, and work with the mathematics content in a new way. Issues that arose during the creative writing class could be addressed while working mathematics problems, and issues that arose in math class became topics for creative writing.

Equally important, the class was a group of 11 women doing mathematics together for six weeks. The authors believe that some of the trust and openness achieved in the writings and discussions would not have occurred as easily in a coeducational atmosphere.

The 1987 pilot project was intended to help previously unsuccessful mathematics students shift their relationships to mathematics. When fearful attitudes interfere with learning, it is especially critical to address those attitudes directly. However, the value of examining mathematics from the perspective provided by the creative writing project may not be limited to math-anxious students. Future projects will include students who are not fearful of mathematics.

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OVERCOMING CONCEPTUAL OBSTACLES: THE EFFECTIVENESS OF CONFLICT DISCUSSION IN THE MATHEMATICS CLASSROOM.

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Abstract

Conflict discussion strategies were communicated to teachers effectively and produced significant improvements in children's understanding. Teachers and students were enthusiastic and stated that the approach benefitted the learning process.

INTRODUCTION

Recent research has indicated that students' conceptual understanding is poor, and most errors are neither random nor careless (Brown et al., 1988; Hart, 1981). Intuitive strategies, used correctly in simple situations, often prove inadequate for more complex questions (Bell, Swan, & Taylor, 1981). If children are to develop superior strategies situations must be devised which demand the use of such strategies as students will only recognise the necessity for change once it becomes obvious that they are using faulty methods (Inhelder, Sinclair & Bovet, 1974). To facilitate conceptual understanding misconceptions must be effectively diagnosed and suitable strategies and materials developed.

An introduction to the concept 'rates', (intensive quantities which are the quotient of two extensive quantities), was considered a suitable topic for examination of such issues with 12- to 14-year-old students. The diagnosis of rate misconceptions was addressed at PME X1 (Bell &

Onslow, 1987); consequently, this paper will examine a strategy for overcoming diagnosed errors by addressing two questions:

i) How effective is teaching based on conflict discussion?

ii) How well can the methodology be communicated to others and used in a meaningful and effective manner?

DESCRIPTION OF THE LEARNING SEQUENCE.

The research was conducted within a Piagetian framework where students were encouraged to face cognitive conflict positively and make constructive use of errors.

The initial activity was short, setting the scene for conflict resolution. Students individually answered a question which, from previous diagnosis, was known to elicit misconceptions. These questions must be at the correct level so as to be neither trivial nor frustrating. Answers were written on the chalkboard, and the children grouped heterogeneously to critique the responses. Consensus was encouraged and rationales recorded by a spokesperson.

Then followed a large class conflict discussion, during which groups shared their findings. When disagreement arose the teacher managed a discussion through which students were encouraged to resolve misconceptions for themselves. Active participation rather than passive acceptance provided the means for students to develop rational arguments and recognise the strengths and deficiencies of different strategies. The classroom climate and teacher's role are crucial. The climate must be one of mutual respect - disagreement with dignity. The teacher must act as a chairperson: asking questions, probing answers, demanding justifications and occasionally playing 'devil's advocate'. It is also the teacher's responsibility to summarize the correct resolutions as succinctly as possible.

The final part of the lesson is one of consolidation. Students need time to reflect on their experiences by examining a few select questions from different perspectives, not by answering many similar examples. These questions would often provide feedback to the earlier conflict situation.

METHOD

Sample

Nine classes and seven teachers from four British schools participated in the experiment. The majority of children were 13 and 14 years of age, and were considered by educational administrators to be representative of a random sample. A total of 157 students completed all aspects of the study.

Procedure

Prior to the start of the experiment a brief meeting was held with each teacher to outline the rationale underlying the experiment. The brevity was deliberate to allow judgement of the feasibility of communicating this methodology through teachers' notes.

All teachers taught one class using the conflict discussion approach. For comparison, two teachers also followed an exemplary style of expository teaching with parallel classes, using activities similar to those found in popular math programs.

The teaching, under both conditions, consisted of six, one hour lessons. Each class was observed for a minimum of 4 hours to determine if the approach adopted in the classroom followed the philosophy outlined in the teachers' notes.

All students took a pre- and posttest. After eleven weeks, during which time they received no further instruction on the topic under investigation, a delayed posttest was administered. To compare the performance of students participating in the experiment with average yearly growth, the same test was administered to 597 eleven to fifteen-year-old students. Where possible, students were chosen from those schools participating in the experiment. Additional students were selected from schools with similar populations.

On completion of the experiment, teachers and students answered a questionnaire designed to assess the strengths and weaknesses of style and materials.

ANALYSIS OF THE DATA

To consider the difference in performance of each class after treatment, the mean scores under the conditions of pre and posttesting were examined. A one-tailed t-test examining differences between two means for correlated samples was used for this purpose. All classes using the conflict discussion approach made significant gains ($p<.001$). Smaller yet still significant gains ($p<.05$) were made by classes using the expository approach.

The two teaching styles can be compared by examining the means and standard deviations of each group (see Table 1). The greater increases of the conflict discussion classes suggest that this method was more successful. Before these gains can be compared reliably, however, the initial differences of the two groups must be taken into account. The multiple regression program PMMD*SMLR (Youngman, 1976) was used for this purpose. A correlational

Table 1 Test Performances for the Two Teaching Groups

Teaching style		Test scores (in %)		
		Pre	Post	Delayed
Conflict discussion	mean	35	56	55
Group (N = 123)	s.d.	14	15	16
Exposition	mean	38	45	46
Group (N = 34)	s.d	13	14	15

analysis (program PMMD*CATT, Youngman, 1976) was then applied to these residual scores to detect differences. The results indicate that the gains made by the conflict discussion classes were significantly superior to those using the expository approach (see Table 2).

Table 2 A Comparison of the two teaching strategies, taking into account the initial differences of mean scores.

Conflict discussion v. Exposition	Pre - post	Pre - delayed
Bartlett Variance (should be >0.05)	0.64	0.76
F-test significance	<.01	<.01

A comparison was made between the teaching groups and students' average yearly gains. Students of the same age as those in the experiment (age 13 and 14 years) averaged 36%, approximately the same as the pretest scores of the teaching groups. Students in the next year group (age 14 and 15 years) achieved an average of 38%. The scores of 11-years-olds were only slightly lower - 32%. Improvement in the conflict discussion groups averaged 20%. Thus, although the delayed posttest means in the teaching experiment were lower than one might desire, it appears that rate concepts are extremely difficult for students to comprehend. Conflict discussion , however, was able to improve understanding significantly.

The main advantage of conflict discussion, as expressed by participating teachers, was that by exposing misconceptions, both they and their students were better able to recognize errors and develop a deeper understanding of the concepts. Examining a few questions in depth rather than following algorithmic approaches led to higher levels of thinking. As one teacher explained: "The children actually begin to think about what they are doing, and reasoning with themselves." Concerns were with management and discussion techniques such as inadequate questioning strategies, belabouring a point or the tendency of a few students to dominate the discussion.

All teachers using conflict discussions stated that they would continue to use this style, and 86% of the students said that they found the conflict discussion approach interesting and beneficial.

CONCLUSION

The findings indicate that conflict discussion is a valuable strategy for improving the level of understanding of children experiencing difficulty with rate problems, and it assists the construction of more complete conceptual frameworks.

Teachers generally became convinced about the conflict discussion approach following a successful lesson. Teaching notes did not always provide sufficient explanation of the underlying philosophy, however, and two teachers needed support and encouragement from the research team in early lessons. Once their students became involved in a stimulating discussion, teachers realized the severity of the misconceptions and were impressed with the desire and enthusiasm generated by those students who, having recognised conflict, were attempting to overcome it. As one of the teachers expressed, "...(discussion) gave insight into how children think and how deep-rooted their misconceptions really are.".

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"ASK THREE BEFORE YOU ASK ME": A TEACHER'S ATTEMPT TO INCREASE THE ROLE OF PEER INVOLVEMENT IN LOGO LEARNING

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Abstract: The inquiry looked at the role played by 10-11 year old classmates as they intervened to help partners at their Logo (graphics) work in an elementary school classroom computer lab environment. Attention to the geometry aspect of the Logo learning was focal. Findings suggested that the relationship between the help-givers and the help-receivers was more collaborative than didactic.

Non-Logo (Cooper, Marquis and Edward, 1986; Forman and Cazden, 1985; Webb, 1982) and Logo (Hawkins, et al., 1982) research studies dealing with the social context of learning have raised questions which gave rise to the present inquiry. Cooper, Marquis and Edward (1986) defined two prototypic situations in which children can learn together: the collaborative and the didactic (1986, p. 270). In collaborative, symmetrical, "doer-doer" relationships between peers, the children have an equal amount of information concerning a problem. In asymmetrical, "knower-doer" relationships, one child knows the information, while the other child does not. Forman and Cazden (1985) have reported that support from a partner seemed to "enable two collaborators to solve problems together before they [were] able to solve them alone" (1985, p. 341). Inherent in both the Forman and Cazden (1985) and the Cooper, Marquis and Edward (1986) studies is the Vygotskian notion of the contribution which a peer can make to another peer's learning. A phrase coined by Thyberg (1985), and heard increasingly often within the community of Logo teachers-- "Ask three before me"-- reflects a belief in peer expertise. This inquiry was undertaken in order to investigate the ways in which the peer classmates intervened to help.

Within the naturalistic elementary school computer laboratory setting of the study, the researcher has looked at both the interaction between

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peer partners, and at the interaction between the partners and other classmates which occurred while the ten to eleven year-old children, in self-selected pairs, were learning Logo. The aspect of peer collaboration and peer teaching between partners and its relationship to the learning of Logo geometry represents one component of the researcher's doctoral thesis (Zack, in preparation). However, the focus of this paper is not on the partners but rather on the interaction which transpired when help was needed by a pair, and consequently neighboring peers intervened to help that pair at their work. Particular attention was paid to the requests for help which were related to the learning of geometry and mathematics concepts (Zack, 1986; Zack, 1988).

RESEARCH DESIGN

The setting was a private, multilingual Jewish day school. All participants (11 girls, 12 boys) came from a middle class background. In 1985-86, the heterogeneous grade five class of 23 children, split into two groups, attended twelve 50-minute Logo computer sessions in the computer classroom equipped with 6 Apple IIe microcomputers, Apple Logo I software, and one printer. The expert Logo teacher, Monica Shapiro, used an individualized approach with the student pairs. The projects were child selected. No changes were made to the Logo software (exceptions: use of slowturtle in grade 3; addition of a HELP command to the startup aids to help students check turtle's heading when needed). Monica wanted the children to learn the Logo theorems via exploration, and via her input when her help was solicited; but the learning of mathematics concepts via Logo was not the primary objective in her agenda. The extent to which the children perceived that Logo offered an environment conducive to the learning of dynamic geometry concepts varied, as became clear in the interview sessions.

The researcher was a non-participant observer. An observational-clinical research design was used. The data included: the researcher's, the teacher's, the children's notes; interviews with the Logo teacher; three clinical interviews with the children (one at the start, one at the close of the grade five session, 1985-6, with 23 children; one at the end of the grade six year, May 1987, with 56 children); and in-depth videotape records, transcribed, of the work of 8 children -- 4 focal children and their partners (both the Logo work and the camera record of their interaction). A separate project was done (Zack, 1986) in order to determine what the children were learning about the Logo geometry concepts, in particular the notion of angular rotation. The researcher's thesis currently in preparation looks at the relationship between the partners and the ways in which the children can contribute to each other's learning of Logo geometry. The present inquiry dealing specifically with intervention by neighboring peers sought to distinguish whether the classmates were contributing to the Logo geometry learning of the 8 children, and if they were, then how were they doing so.

The transcript data representing the twelve weeks of work of four children and their partners were reviewed; instances of solicited and unsolicited peer interventions relating to the 8 children were tallied and analyzed to determine (a) the circumstances in which the students would ask a neighboring peer for help; (b) the frequency with which the various types of problems were referenced, that is, mathematics and geometry concepts, syntax, plan and programming problems; (c) the children's response to solicited versus unsolicited peer interventions; (d) the degree to which the peer help was indeed "helpful"; (e) the instances in which the helping episode could be considered a teaching episode; and (f) the strategies used by the child-teachers (for example, guiding, demonstrating, using commentary, giving meaningful explanations).

FINDINGS

Solicited and unsolicited interventions

The partners almost always attempted to work through a problem themselves first. Once they decided that they needed help, their first reaction tended to be to elicit help from the "expert" source, that is, the Logo teacher, Monica. They would elicit help from neighboring peers when the teacher was busy. That the first choice of call in this setting would be the Logo teacher was not surprising. It is clear from the interview data that some children considered it the teacher's role to teach; the children frequently stressed that the teacher was the expert--and this teacher was indeed an expert in the Logo area. Seven of the twenty-nine solicited peer interventions from classmates were the result of explicit redirections by the teacher. The teacher would when busy direct the petitioner to "ask somebody else," or would use a variant f Thyberg's phrase: "Ask three before you ask me." Unsolicited offers of elp by a peer classmate were on the whole not well received. Offers to elp were refused, and stated suggestions by would-be contributors were gnored, in 13 of the 16 unsolicited interventions.

Calls for help with "syntax" problems (those related to working the omputer functions such as loading, saving, editing) outnumbered calls for elp with mathematics and geometry problems by a ratio of 4 to 1. The earning was considered to be in an early phase (Zack, 1988); consequently he children were still struggling with controlling the machine workings. hereas the Logo geometry aspect was more forgiving, in that the partners ould "fiddle around" with angle and distance and arrive at a somewhat lose approximation, the syntax required EXACT specifications; hence erhaps the large proportion of calls both for teacher and peer ntervention concerning syntax problems as opposed to geometry and athematics problems.

iving help

It was found that the peer interventions were as often incorrect as hey were correct, except in the case of one specific child (Russ). The ajority of the solicited and unsolicited interventions were executed by hildren who were themselves still at the exploration stage, and so the elationship between giver of help and receiver of help was on the ollaborative (Cooper, et al., 1986) level, rather than on the didactic Cooper, et al., 1986) level. This finding is consistent with the pattern f working relationships found to exist between the partners themselves in he larger study, that is, the relationships were found to be ollaborative ones, except in certain instances where Russ was involved as partner (Zack, in preparation). This finding in relation to ollaboration is also consistent with that of Hawkins, et al. (1982) who eported that there were few peer teaching episodes, and a far larger roportion of collaborative interactions in their study.

In the "knower-doer" relationship, the peer expert knows the nformation. There was one child, Russ, who was considered by the esearcher to be a "computer expert" in that he was in command of the Logo yntax, geometry and programming concepts; his interventions were correct n all but one instance. Three self-proclaimed "computer experts" were een to emerge; they offered help, but their suggestions were often nappropriate and incorrect.

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Peer teaching skills

There was a range in the ways in which help was administered. On six occasions, the peer helper performed the function without commentary, an approach which was not conducive to learning, as the children themselves noted in the interviews. However, in the majority of peer interventions, some teaching skills--such as focusing, guiding while the peer performed, giving information, demonstrating with commentary--were in evidence. In one instance, the above-mentioned "class expert," Russ, was specifically assigned to the role of tutor by the teacher; he was asked to help two students with the mathematics concept of variable. In that designated tutoring session, the "knower's" peer status changed, the time frame became more extensive, and more teaching functions emerged than was the case in the other peer interventions; these teaching skills included--checking on progress, marking critical features, giving explanations, and eliciting critical thinking.

Interventions related to mathematics concepts

The breakdown of the 9 peer interventions related to mathematics and geometry concepts revealed that 4 were concerned with angular rotation, 3 with circle and semi-circle, 1 with variable, and 1 involved a check on the accuracy of a calculation. There were three unsolicited peer evaluation statements related to errors in angular rotation, in which classmates commented on the "crooked" or "jagged" look of another pair's line but gave no suggestion as to what the partners might do to correct the error. The "jagged line" was the result of an input or 85 or other, when 90 should have been keyed in. None of the "onlookers" were heard to offer a suggestion for change. In each case the evaluative comment was ignored or dismissed by the receivers (Zack, 1988). In accordance with Webb's (1982) finding that "receiving help was effective only when given in response to [expressed] student need (1982, p. 425), and Webb's observation that "receiving explanations about the task would not be expected to relate to achievement if the explanations are inaccurate" (1982, p. 427) or if no explanation at all is given (1982, p. 425), one would not expect that in the instances of the unsolicited evaluations, or in instances of incorrect help, learning would result. However, there were indications in this study that Webb's criteria of "correctness" and "explanation" may be more consistent with the didactic than with the collaborative situation; as will be described below in the episode between

Saul and Shauna, a partial suggestion without attendant explanation, when given at the appropriate moment, was seen to spur learning.

Two helpful interventions

There were two helpful interventions, one related to turtle heading, the other related to doing the inverse of a curve. What was most interesting was that neither intervention was complete, nor answered the need fully. But each was nevertheless helpful to the petitioner. In the first case, after a false, incorrect start, Saul, one of the three afore-mentioned self-declared experts, gave Shauna the nudge she needed to discover that she could use FD X BK X when turtle's pen was up, as a tool for checking turtle's heading. Without the collaboration of Saul who gave Shauna the beginnings of an idea of how to proceed toward the solution, Shauna could not have gone on to re-invent the HELP routine for herself. This episode was a prime example of how "doers," in interacting together, were able to contribute to the development of a product which one child could not have accomplished on her or his own (cf. Forman and Cazden's (1985) above-mentioned comment, but in this case in relation to classmates, not to working partners).

In the second case, Jon and Eric sought help from the teacher to enable them to delete a curve by doing the inverse. They could not solve the problem of doing a two step inverse, that is, since they had keyed in REPEAT 30 [FD 5 LT 5], what they needed was REPEAT 30 [RT 5 BK 5] to undo. The teacher redirected them to ask a peer. David, the able Logo user solicited by Jon and Eric, could not help solve the problem of doing the inverse of REPEAT 30 [FD 5 LT 5], so instead he showed them how to circumvent the problem by going HOME and starting again. Thus Jon and Eric did not learn the geometry aspect from David, but they did solve their problem!

The Logo teacher's reflections upon "Ask three before you ask me"

During the researcher-teacher interview (June, 1987), Monica was surprised to learn that the tapes revealed such a large number of peer interventions (48 in all), for it was her impression that the children had rarely asked neighboring peers for help, despite her encouragement to them to do so. She was also surprised to find that a substantial portion of the peer interventions were not correct or helpful. These findings caused her to reflect upon the complex issues underlying the seemingly simple teacher exhortation to "Ask three before you ask me." One of Monica's suggestions was that despite the fact that the inability to obtain expert help may result in frustration for the soliciting child, nevertheless part of the learning process in this peer-centered collaborative setting entailed developing the skill of distinguishing good help from incorrect help.

Conclusions: The children issue a "call for service" (Marilyn Merritt) with a view to receiving expert and correct help (from a teacher or from an able peer) in order that they might get on with their work. However, in this study the directive to "Ask three before you ask me" was less a matter of "recognizing and promoting peer expertise" (1985, p. 1) than Thyberg expressed in her article. It is, rather, more a matter of the children learning to (1) acknowledge that the child helper may herself need collaboration; (2) develop the skill of differentiating the "good" help from the "bad" help; and (3) persevere at times with the quest until the appropriate help and expertise is found.

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INTEGRATING RESEARCH ON LEARNING AND INSTRUCTION

A Symposium Presented by Affiliates of the
National Center for Research in Mathematical Sciences
Education

Participants

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Thomas P. Carpenter, University of Wisconsin-Madison
Catherine A. Brown, Virginia Technical Institute
Penelope L. Peterson, Michigan State University
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Thomas R. Post, University of Minnesota
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OVERVIEW

In the last ten years, significant advances have been made in understanding children's cognitions in particular mathematical content. Also, we know a great deal about instructional processes and the importance of considering the teacher as a thoughtful decision maker. However, until recently there has little research which integrates knowledge from these two bodies of work. One of the major goals of the National Center for Research in Mathematical Sciences Education is to facilitate research which integrates research on teaching and research on learning. In order to accomplish this goal, people who are interested in research on teaching and/or learning have been asked to affiliate with the Center and to plan and conduct research which considers both teaching and learning mathematics. The purpose of this symposium is to provide a forum for affiliates, and those who wish to become affiliates, to engage in dialog about theoretical issues, important research questions, and appropriate methodologies and procedures for implementing integrative research on teaching and learning.

In order to initiate the dialog, theoretical statements, completed research, or plans for research will be presented by various panel members. A brief synopsis of each presentation follows.

Elizabeth Fennema, Chair "Integrating Research on Teaching and Learning Mathematics: An Overview". Briefly reviewed will be major findings from relevant teaching and learning research. The importance of integrative work will be explored and the purposes of the National Center presented.

Thomas P. Carpenter. "Characterizing Integrative Research on Teaching and Learning Mathematics." Two major approaches to integrating teaching/learning research will be presented and contrasted. Major assumptions underlying the work which has been done at Wisconsin by Carpenter, Fennema and Peterson (In Progress) will be identified, compared, and contrasted with major assumptions of work done at Purdue by Cobb, Yackel, and Wood. Similarities and differences will be discussed.

Alba Thompson. "Contributions of Previous Work on Teachers' Beliefs to Integrative Work on Teaching and Learning." Explored will be how work which has previously been reported on teachers' beliefs about mathematics and mathematics teaching, can contribute to integrative research. Implications of previous research which should be considered in planning integrative research will be presented.

Penelope L. Peterson. "Teachers' Cognitions and Research on Teaching and Learning". Identified from previous work will be relevant issues which should impact on integrative research. Theoretical issues concerning studying teaching and learning will be presented for discussion.

Catherine A. Brown. "Learning to Teach Mathematics" Presented for discussion will be plans to study novice middle school teachers' subject matter knowledge and beliefs about mathematics and mathematics teaching. Incorporated will be discussion of how knowledge about students' learning of rational numbers has had an impact on the study.

Thomas R. Post. "Teachers' Knowledge Structures and Beliefs about Rational Numbers". Teachers often have inadequate knowledge of the mathematics they are expected to teach. Questions will be raised about how the knowledge that teachers have should be considered as integrative research is planned.

Diana Wearne. "Studying the Application of Teaching/Learning Research to Classroom Instruction." Explored will be how integrative research on teaching and learning can be translated to classroom instruction. A research paradigm for studying the effectiveness of this translation will be presented.

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COGNITION, METACOGNITION, AND THE VAN HIELE MODEL OF THINKING IN GEOMETRY

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ABSTRACT. Research on the van Hiele model of thinking in geometry focused initially on the model's usefulness for characterizing the thinking of elementary and secondary school students. More recent research on the model has involved various contexts such as Logo and geometry learning and, more generally, in relation to aspects of developmental and cognitive psychology. The symposium is designed to stimulate interaction and discussion among the presenters and those attending about the van Hiele model in relation to some aspects of cognitive psychology such as the representation of knowledge and the role of metacognition in learning. As a catalyst for this discussion, the symposium includes videotaped episodes of elementary students doing van Hiele-based activities in clinical interviews and in Logo-based classroom environments and a demonstration of a software tool designed to promote students' development of geometric knowledge and metacognition.

INTRODUCTION

During the past ten years there has been considerable research in the United States and abroad on the model of levels of thinking in geometry formulated by the Dutch educators, P. M. van Hiele and his late wife Dina van Hiele-Geldof (van Hiele, 1957/1984; van Hiele, 1959/1984; van Hiele, 1986; van Hiele & van Hiele-Geldof, 1958; van Hiele-Geldof, 1957/1984). According to the van Hieles, the learner, assisted by appropriate instruction, passes through levels of thinking, from recognition of shapes by their appeareance (level 0) to analysis of shapes in terms of their properties (level 1) to three higher levels involving informal deductions (level 2), then deduction in an axiomatic setting (level 3), and finally rigorous work with abstract geometric systems (level 4). Features of the model include the hierarchial nature of the levels, differences in objects of thought and in language at each level, and an intrinsic-extrinsic relationship between ideas on each level. The van Hieles assert that one cannot skip levels and that progress from one level to the next is more dependent upon instruction than on age or biological maturation.

Initial research on the model during the early 1980's focused mainly on validation of features of the model (e.g., the hierarchial nature of the levels, discontinuity between levels) and on the appropriateness of the model for characterizing the geometrical thinking of elementary and secondary students (Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1985, 1988; Usiskin, 1982) and teachers (Mayberry, 1983). More recent research has investigated the model in contexts such as Logo and geometry learning (Battista & Clements, 1987; Ludwig & Kieren, 1986; Olive & Scally, 1984; Scally, 1986), achievement testing (Pedersen, 1987), and textbook material on geometry (Fuys, Geddes, & Tischler, 1985, 1988; Joyce, 1984). However, there have been only a few studies on the model in relation to broader psychological contexts such as from Piagetian (Denis, 1987; Orton, 1987) and cognitive psychology (Lehrer, 1988) perspectives.

OVERVIEW OF SYMPOSIUM

This symposium is intended to foster interaction and discussion among the presenters and those attending about the van Hiele model in relation to some important ideas in cognitive psychology -- in particular, involving cognition (such as representation of knowledge, declarative and procedural knowledge) and metacognition (such as knowledge of cognition and regulation of cognition). As stimuli for discussion and questions, the presenters will show several videotaped episodes of students (grades 5 to 9) engaged in van Hiele-based geometry activities and will demonstrate a software tool for exploring informal geometry.

The first part of the symposium will feature video segments of sixth and ninth graders in the Brooklyn College van Hiele Project (Fuys, Geddes, & Tischler, 1985, 1988) as they work one-to-one with an interviewer on three instructional-assessment modules (Properties of Quadrilaterals, Angle Sums of Polygons; Area). The modules were designed to assess in a dynamic way a student's level of thinking based on the student's responses to key questions and activities at the start of the modules and throughout each module. Responses were analyzed in light of Project-developed behavioral descriptors for the van Hiele levels. Over the six to eight interview sessions most students made progress within a level and to a higher level; some even progressed toward level 2 by following and giving informal deductive arguments. Results also indicated that there appeared to be change in a students' metacognition, in particular about explectations in activities and monitoring of their thinking. This suggests that metacognitive level descriptors be included with the cognitive ones usually associated with the van Hiele model, thereby making more explicit provision for metacognition in the model. Discussion of the video segments will focus mainly on metacognitive apsects of students' thinking on key tasks such as sorting cutout shapes, listing properties for a class of figures, establishing subclass inclusions, and using family trees in informal deductions. Metacognitive knowledge (e.g. with respect to persons, tasks, strategies) and regulation of cognition (Flavell, 1979; Garofalo & Lester, 1985) will be examined in this van Hiele context along with the role of the interviewer in assisting students to monitor their thinking. Also, some other aspects of learning from a cognitive psychology perspective (Shuell, 1986) will be considered: the ways knowledge can be represented at van Hiele levels 0, 1, and 2; the active, goal-oriented, constructive nature of learning as seen in the "teaching experiment" or "instructional" components of the interviews; and the role of domain specific knowledge and learning skills, in particular of students on key tasks that initiate them to a higher level of thinking and of the interviewer when functioning as an "expert" teacher during the clinical interviews.

This first part of the symposium illustrates ways that mathematics educators can draw upon ideas from cognitive psychology to focus, refine, and extend analyses of a student's performance on van Hiele-based tasks. Bridges between research in mathematics education and cognitive psychology can also be built by cognitive psychologists who use geometry learning situations as contexts for researching constructs in cognitive psychology, as indicated by the second part of the symposium. The second part focuses on how software and teacher contexts can influence fourth and fifth graders' cognition and metacognition. This part is thematically identical to the first, but videotaped episodes in this part feature Logo-based instructional settings designed to foster transition between van Hiele levels. The design of the software, which will be demonstrated, reflects a van Hiele-based analysis of informal geometry in that it promotes connections between visual features, properties, and relationships among properties of simple Euclidean figures. The development of the software as a learning tool with cognitive and metacognitive outcomes was in part an outgrowth of previous research (Lehrer, Guckenberg, & Sancilio, in press) that explored fourth graders' development of declarative (i.e., verbal propositions) and procedural (e.g., a Logo program) interpretations of geometric concepts and of their "pre-proof" thinking at van Hiele levels 0, 1, and 2. Results indicated how these develop in a dynamic, interrelated way through small-group work on "carefully crafted activities" based on the levels. Parallels were found between the development of children's organization of geometric knowledge from a cognitive psychology perspective and the first three van Hiele levels. The newly developed software and teacher interventions provide a context for further development of students' metacognition as well as geometric knowledge. An overview of this on-going research will be presented along with indications for further related research in cognitive psychology.

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AN ASSESSMENT INSTRUMENT TO EXAMINE KNOWLEDGE OF MULTIPLICATION AND DIVISION CONCEPTS AND ITS IMPLEMENTATION WITH INSERVICE TEACHERS

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In this paper we will present an instrument which we have developed to assess knowledge in the domain of multiplication and division problems and discuss its theoretical background. The instrument controls five variables which have been identified as influencing the solution of arithmetic problems, and it has been used to trace difficulties, misconceptions, and solution processes possessed by inservice elementary school teachers. The main recommendation from this assessment is that transitional stages from the intuitive notion to the formal notion of multiplication and division concepts must involve the concept of proportion. This concept would enable students to represent "non intuitive" multiplication and division problems as missing value proportion problems and to apply proportional reasoning strategies to solve these problems.

Fischbein, Deri, Nello, and Marino (1985) argued that conception and performance of multiplication and division problems are unconsciously derived from three intuitive models. The model associated with multiplication problems is repeated addition. Under this model the roles played by the quantities multiplied are asymmetrical (Greer, 1985). One of the quantities multiplied, called multiplier, is conceived of as the number of equivalent collections, while the other quantity, called multiplicand, is conceived of as the size of each collection. These conceptions impose the constraint that multipliers must be whole numbers, thus they reinforce the misconception that products must be larger than multiplicands, or "multiplication makes bigger." For division, Fischbein et al., (1985) suggested two intuitive models, one is associated with

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equal sharing, or partitive division problems, the other with measurement, or quotitive division In the partitive division model, an object or collection of objects is divided into a number of equal fragments or subcollection Associated with this model are the constraints that divisors must be whole numbers and smaller than dividends; these constrainte result in the misconception, or constraint, that quotients must be smaller that dividends, or "division makes smaller." The quotitive division model is associated with division problems in which it is required to find how many times a given quantity is contained in a larger quantity The only constraint imposed by this model is that divisors must be smaller than dividends

Fischbein et al., (1985) addressed the question concerning the conflict between the intuitive models and the formal operations of multiplication and division with 5-th, 7-th, and 9-th graders. Tirosh, Greaber, and Glover (1986) addressed the same question with student teachers, and further investigated the similarity between children's errors and student teachers' errors in solving multiplication and division problems. However, in these studies only one variable was controlled, the number type variable (i.e., whole number, decimal > 1, decimal < 1), leaving other variables, such as the textual, contextual, and syntactic variables uncontrolled. Mangan (1986) addressed these questions in a study which controls systematically the textual and the number type variables, with extensive age range of children and adults: primary and secondary school students, further education, university students, and student teachers

The instrument we have developed to investigate questions related to multiplication and division concepts is different from those used in these studies, in the number of confounding variables it controls The instrument consists of a written, short answer problems followed by one-on-one interview assessment, where the problems used were constructed by controlling five variables, the textual, structural, contextual, numerical, and syntactical variable.

Textual variable. In analyzing the textual structure of multiplicative problems, Nesher (1988) identified two subcategories within the isomorphism of measures category (Vergnaud, 1983): mapping rule (e.g., "There are 5 shelves of books in Dan's room. Dan put 8 books on each shelf. How many books are there in his room?") and multiplicative compare (e.g., "Dan has 12 marbles. Ruth has 6 times as many marbles as Dan has. How many marbles does Ruth have?"). In analyzing these categories we found that there is a fundamental difference between them (Harel, Post, and Behr, 1988). While in multiplication mapping rule problems the roles of the quantities multiplied--as multiplier or multiplicand--can be directly derived from the problem statement, in multiplication compare problems, these roles are interchangeable, depending on the solver's interpretation of the phrase "times as many as." Accordingly, a division compare problem can be represented either as partitive or quotitive depending upon the solver's interpretation of the roles of the quantities in the corresponding multiplication problem. This implies that the theory of Fischbein et al., (1985) cannot be examined on multiplicative compare problems, since their structure cannot be determined prior to testing. Our instrument addresses two aspects concerning the textual variable. First, it tests Fischbien's theory with problems whose textual structure is that of the mapping rule type. Second, it tests the solver's representations of problems with the multiplicative compare structure (for more on the latter aspect, see Harel et al., 1988).

Problem structure variable. The instrument includes the three

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problem structures, multiplication, partitive division, and quotitive division.

Context. The context of the problems used is that of rate. The only rate problems included are of the following types: weight, price, and consumption.

Number type. The instrument tests knowledge of multiplication and division concepts in the two domains of the number system, decimals and fractions. The problem quantities--multipliers and multiplicands and divisor and dividend--were systematically varied across number type, whole number, decimal > 1, and decimal < 1, and across the order relation between problem quantities. In addition, the instrument includes problems in which the quantities varied across whole number, fraction > 1, and fraction < 1.

Syntactic. This variable is derived from the analysis by Harel and Behr (in press). It involves the subvariables "location of the missing value" and "coordination of measure spaces." (See Conner, Harel, and Behr, this volume)

IMPLEMENTATION

As has been said earlier, our instrument is distinctive from others' because it controls a wide range of confounding variables. In addition, the instrument has been used to examine the multiplication and division knowledge of a different population, inservice elementary school teachers. The objective was to assess teachers' knowledge, with respect to the following questions: (a) do inservice teachers solve correctly formal problems (i.e., problems which conflict with the intuitive models)?; (b) what strategies do teachers use to solve formal problems?; (c) to what extent the teachers' errors in solving formal problems are similar to those of children?; (d) to what extent the solution processes used by teachers to solve multiplication and division problems are similar to those used by children?; (f) what impact does the textual structure of multiplicative problems have on the problem representation and problem difficulty?; (e) are the intuitive rules equally robust in solving multiplicative problems? The latter question needs more elaboration; it will be given in the following section.

Levels of robustness of the intuitive rules. A careful examination of different studies led us to conclude that the intuitive rules are not equally robust in problem solutions. Consider, for example, the table below which shows the percentage distribution of responses to problems 16, 17, 20, 21, and 22 from Fischbein et al.'s (1985, p. 12). All these problems are of partitive division type which violate the same intuitive rule, "divisor must be smaller than the dividend." Despite this uniformity, the results are strikingly different: the percentages of correct responses on Problems 16 and 17 are much lower than of those on Problems 20, 21, and 22. Fischbein et al.'s explanation to this is that in Problems 16 and 17 the students' tendency was to reverse the roles of the numbers as a divisor and dividend, "had they done that in Problems 20 to 22, however, they would have ended up with a decimal divisor! It appears that, faced with having to cope with a violation of the partitive model's rules, the pupils chose instead not to reverse the numbers" (p. 13). From this result we concluded that different intuitive rules within the partitive model may not be equally robust in solving partitive division problems: in Problems 20, 21, and 22 the children preferred to cope with the rule, "divisor must be smaller than dividend," than with the rule "divisor must be a whole number."

Problem	Operation	% Correct (Grade)
16	5÷15	20 (5), 24 (7), 41 (9)
17	5÷12	14 (5), 30 (7), 40 (9)
20	3.25÷5	73 (5), 71 (7), 84 (9)
21	75÷5	85 (5), 77 (7), 83 (9)
22	1.25÷5	66 (5), 74 (7), 70 (9)

The question of how different rule violations effect differently the solution of multiplicative problems needs further investigation. As a first step, we classified multiplicative problems according to the types of rule violations Consider, for example, the partitive model, which was characterized by the three rules (a) divisors must be whole numbers; (b) divisors must be smaller than dividends; and (c) quotients must be smaller than dividends Based on the type of rule violation, partitive division problems can be classified into six categories: PV(0), PV(a), PV(b), PV(a, b), PV(a, c), and PV(a, b, c), where P stands for partitive, V for violation, and a, b, and c for the rule being violated (0, when no rule is violated) Thus, PV(a), for example, is the set of partitive division problems which violate Rule (a). Using this classification, we were able to identify a complexity hierarchy among multiplication and division problems, and to suggest models for the cognitive processes involved in solving these problems.

RECOMMENDATIONS

Fischbein et al. (1985) believe that the intuitive models correspond to features of human mental behavior that are primary, natural, and basic" (p. 15). Acknowledging the fact that these models are behaviorally and intuitively meaningful, teachers use them to introduce the operations of multiplication and division; thus they reinforce the conflict with the formal concepts of these operations (p. 15-16). To resolve this didactical dilemma, several recommendations were offered. Fischbein et al., recommended that teachers should "provide learners with efficient mental strategies that would enable them to control the impact of these primitive models" (p. 16). Greer (1985) chose a different direction he recommended that teachers should "aim to widen the range of models available to the pupils" (p. 74). However, neither Fischbein nor Greer specified his recommendation. Our recent work with the multiplicative field and results from the assessment of inservice teachers' knowledge suggest that the mental processes involved in the comprehension and solution of the intuitive problems (i.e., problems which conform to the intuitive models) are fundamentally different from those involved in the formal problems (i.e., problems which conflict with the intuitive models). As such, transitional stages from the domain of intuitive problems to the domain of formal problems are required. Our conclusion is that these transitional stages must include the concept of proportion, which can enable learners to represent formal problems as missing value proportion problems. Intuitive solution strategies are available to this representation. These strategies involve determining the multiplicative relationship between two given quantities and extending that relationship to the other two quantities to find the unknown quantity (see Harel and Behr, in press; Vergnaud, 1983). Primary data with inservice and preservice elementary school teachers and results from work with 7-th graders (see Sellke, Behr and Voelker, this volume) support this approach. However, still more work is needed to find how to integrate the two domains of problems into a one coherent domain within the multiplicative field.

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A MODEL FOR THE DEVELOPMENT OF LEADERSHIP AND THE ASSESSMENT OF MATHEMATICAL AND PEDAGOGICAL KNOWLEDGE OF MIDDLE SCHOOL TEACHERS[*]

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ABSTRACT

Determining the mathematical and pedagogical knowledge of middle school teachers was the first phase of the Mathematics Leadership Project being conducted by the University of Minnesota and Northern Illinois University under a grant from the National Science Foundation. We discuss preliminary results of the teacher profile, an instrument designed to assess rational number multiplication and division understandings of middle schoo teachers. Based on analyses of these data, an inservice mathematics leadership training program was initiated during the summer of 1988. Regular follow-up sessions are scheduled for the 1988-89 school year.

OVERVIEW

The basic goal of the Mathematics Leadership Project is to formulate and test a model for developing professional leadership abilities in intermediate grade (4-6) teachers of mathematics. To achieve this goal, the project was partitioned into three phases:

Phase 1 (September 1987 through March 1988). A teacher profile was compiled and administered. It consisted of two written instruments and an interview instrument (De Kalb N = 51; Minnesota N = 167; 15 teachers were interviewed at each site).

Phase 2 (March through July, 1988). A four-week, full-day staff development institute for teachers of mathematics in grades 4-6, was planned and executed. A significant portion of the agenda resulted from needs identified by th

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[**] The authors wish to acknowledge important contributions made to this project by Anne Bartel, Kathleen Cramer, Sarah Currier, and Nancy Williams.

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preliminary teacher profile in Phase 1.

Phase 3 (September 1988 through June 1989). Leadership and follow-up activities will be conducted. Teachers in the workshop will attend half day twice monthly, or, monthly full day follow-up sessions throughout the school year. Teacher leadership teams will plan and carry out at least one inservice workshop for their colleagues at their building site as well as provide a variety of in-class consultative services. They will also begin to involve parents in the curricular renewal process. Our approach here is systemic in nature; teachers, students, administrators, and parents all will become involved in the change process.

Research findings relevant to the workshop.

As of this writing, data collected from the three instruments administered in Phase 1 of the project have not yet been completely analyzed. However, certain characteristics of teacher perceptions of mathematical concepts emerged early in the analysis.

Teachers lacked a clear understanding of the part-whole/partitioning concept. Specifically, given a model representing a part of a whole, they had difficulty representing the whole. They could generate an area model for a fraction but had trouble representing that fraction when given various sets of discrete objects which defined the whole. In this respect, teacher conceptions of part-whole/partitioning were found to be similar to those of children in the grades they teach (Behr, Post, Lesh, 1986). See also accompanying paper for examples and discussion of fraction test items (Cramer & Lesh).

Teachers had difficulty solving division problems when both divisor and dividend were less than one. This was true whether numbers were expressed in fraction or decimal form. When encouraged to draw a picture to model a story problem involving such division, however, they were often able to solve such problems. Nevertheless, many teachers had great difficulty composing a story problem involving division of nonintegral rational numbers. In fact, when asked to do so, they often wrote a story problem that required the inverse operation of multiplication. In short, teachers held certain primitive models of multiplication and division such as "multiplication makes bigger, and division makes smaller." These tended to dominate their strategies. Similar domination of thought processes by such primitive models was also found in children (Fischbein, Deri, Sainatinello & Marino, 1985).

A disturbingly large percentage of teachers were not successful with relatively routine categories of rational number knowledge. Part I of the teacher profile contained 58 short answer or multiple choice items. Part II contained 6 problems with directions to "solve and to explain how you would describe your solution to children."

Table I below lists the main categories of items which comprised Part I, along with the mean percentage correct by site.

Category	# of Items in Category	Percent Correct Minnesota N = 167	Percent Correct De Kalb N = 51
Part Whole	2	73	67
Ordering Fractions	4	65	65
Ordering Decimals	2	54	73
Fraction Equivalence	4	50	51
Qualitative Approaches	3	38	49
Percent	3	64	76
Operations w/Fractions	13	74	66
Operations w/Decimals	5,3*	92	90
Conversions (Frac. to Dec.)	3,1*	85	63
Concept of Unit	4,3*	64	70
Estimation	5,4*	71	77
Ratio Problems	5,4*	48	56
Compare Rates	3,2*	78	90
Overall Mean		66	69

*The first number refers to Minnesota, the second to De Kalb.

Table I - Teacher Profile - Part I Results
(Entries are percent correct by item category)

Other aspects of the teacher profile dealt with single operation multiplication and division problems with "awkward" fractions and decimals. The instrument related to that aspect of our investigation is discussed elsewhere in this volume (Harel, Post, & Behr).

Although space here does not permit an expanded discussion of these results, a cursory inspection of Table I suggests that achievement levels across sites is relatively stable and that overall approximately 30% of the items were missed by

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teachers who are to various degrees involved in classroom instruction in the majority of these topical areas. A closer inspection of the distributions of teachers scores suggested like percentages of teachers achieving a score of less than 50% on the instrument. Surely this is cause for concern and a topic in need of serious (and constructive) discussion by our profession.

Use of research findings in planning workshop.

At the NIU site, topics to be covered in the workshop were determined by the instructional team on the basis of their past experience in observing mathematical strengths and weaknesses found among local teachers, recommendations for mathematics curricular changes (NCTM, 1987), and the research finding noted above.

The goal of the workshop was to enhance teacher perceptions of certain mathematical concepts by increasing their background in the areas addressed by the profile, place value, multiplication and division of whole numbers, non-negative rational numbers, proportional reasoning, geometry (both 2- and 3-dimensional), probability, statistics, counting principles and elementary number theory. Leadership participants will be expected to consult with other teachers about some of these issues during 1988-89.

The Minnesota site did little with place value, multiplication and division, statistics and number theory. More effort was devoted to part whole, order and equivalence, ratio and proportion. These concepts were taught in a way suitable for use in the classroom; namely, a variety of manipulatives were used to introduce concepts, instruction was learner-centered and discovery oriented, and emphasis was placed on problem solving activities, mental arithmetic, estimation, and the appropriate use of the hand-held calculator.

Because of research findings related to other aspects of the profile, much time was spent in De Kalb developing the concept of division of whole numbers, using both the measurement and partitive models, to ensure sufficient conceptional background for division of rationals. Later, a variety of physical models, both discrete and continuous, were used to develop the concepts of order and equivalence of fractions, and operations with fractions. The units on geometry, counting and number theory, probability and statistics complemented and extended the understanding of rational number. These units also provided numerous concrete examples and problems related to proportionality. Estimating and checking for reasonableness of answers was stressed at both sites.

At Minnesota, translations within and between modes of representation were stressed. Understanding was defined as the ability to make appropriate translations. This has enormous implications for the ways in which mathematical knowledge is assessed and also for the general nature of mathematics instruction. Much more will be done with this idea during the upcoming year.

Participants were encouraged to verbalize and later to write as a group generalizations, rules or algorithms which they had discovered. They were also asked to develop story problems on topics covered, suitable for use in their classrooms.

Four days each week were to be spent in classroom activities at the NIU campus. The fifth day was to be used to develop individual or team teaching-learning units to be used in classrooms in 1988-89. Three units would be required of each team: one in fractional numbers, one in geometry, and one in probability, and/or statistics, permutations, or number theory. These units were to be student centered, including activities using a variety of physical models. The fifth day was to be used also to develop outlines for the workshops that teams would present to their colleagues. These activities were to be carried out at school or community sites, with members of the NIU instructional staff available at various school sites and at the university.

In Minnesota, five days per week were spent on site; however, two to three afternoons per week were spent by teacher teams developing and synthesizing materials for classroom use during 1988-89. This time was coveted and appreciated by teachers and, in retrospect, is felt to be an important element in any such future undertaking.

Leadership Institute--In retrospect.

Participants varied considerably in the breadth and depth of their mathematical knowledge. For example, at NIU, several participants had not mastered the concept of place value and had difficulty identifying a unit. Very few had any knowledge of geometry or had even the most fundamental perceptions of number theory, counting, probability, or statistics concepts. Most participants followed the textbook rigidly and seldom got to the back of the text where non-computational topics were discussed. This not uncommon phenomenon is probably more a function of systemic expectation than it is the result of individual teacher decisions. Very few at the De Kalb site assigned story problems because, "Children don't like them." Most had had little experience in writing story problems and found it difficult to do so. Almost all worked well in their teams of two or three and often spontaneously joined together to form double teams of four or five when appropriate. At NIU, the long car pooling commute that several teams faced each day had the unplanned benefit of uniting teams and providing time for informal problem solving sessions. Although participants varied in their willingness to shed old styles of learning and teaching and to try new ones, all were enthusiastic learners.

At the Minnesota site, about one half of the participants had been involved in a previous one-week leadership institute (summer 1986 or summer 1987) designed primarily to acquaint them with current thinking in the field and to suggest, by example, alternative instructional formats. As such, these individuals were already enthusiastic about prospects for the new project. The teachers were eager to learn about profile results and, in particular, to "fill in" identified conceptual gaps. This was important since it is these individuals who will serve as leaders in the next cycle of this project during the summer of 1989. Phase 3 of the program will begin in September of 1988. It is there that the fruits of these efforts will be assessed.

As mathematics educators continue to attempt to improve pre-secondary mathematics programs, principally by expanding the scope of appropriate components (problem solving, geometry, estimation, computers, etc.), we must realize that we are asking a significant percentage of teachers to teach concepts which they themselves have never (or so far in the distant past so as to be unretrievable) been exposed to as students. Add to this the variety of additional "add-ons," which further complicate the ever expanding role of the elementary school teacher, and the results presented here become perhaps more understandable although certainly not more acceptable. It seems feasible for us to urge consideration of alternative delivery systems as well as alternative school organizational patterns and to carefully identify the implications each would have on the structure and substance of elementary school mathematics programs. We are convinced that teachers are also very interested in substantive progress.

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RATIONAL NUMBER KNOWLEDGE OF PRESERVICE ELEMENTARY EDUCATION TEACHERS[*]

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ABSTRACT

This study assessed the rational number knowledge of 48 elementary education majors. The test used, in part, a subset of items from a testing instrument developed by the Mathematics Leadership Project at Northern Illinois University and University of Minnesota. The paper discusses results of the test and presents sample test items. The data suggest that some 20% of the preservice teachers who took this test do not understand rational number ideas well enough to be expected to teach fractions meaningfully.

Introduction

A subset of the teacher profile developed as part of the Mathematics Leadership Project (see paper in these proceedings by LaCampagne, Post, Harel, and Behr) was administered to preservice teachers in elementary education enrolled in an elementary mathematics methods class at the University of Wisconsin-River Falls. The purpose of the testing was to assess students' conceptual understanding of rational number ideas. Many of the test items reflect content taught to fourth graders participating in the National Science Foundation sponsored Rational Number Project (Behr, Lesh, Post & Silver, 1983). The test assessed content that these preservice teachers will shortly be teaching upon completion of their degree.

Participants

Forty-eight junior and senior level students were given a 45 item fraction test covering the categories listed in Table 1. All but one student had completed two, 3 quarter credit mathematics classes designed for elementary education majors. These two courses met the mathematics requirements for the University.

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Fifty-four percent of the students had taken no other college level mathematics class; 13 percent had taken one or two remedial classes (below college algebra); 29 percent had taken classes ranging from college algebra to calculus; 4 percent had a mathematics minor.

Test Results

Table 1

Results of Fraction Test Given to Preservice Elementary Education Teachers

Category Type	Number of Items	Percent Correct
Fraction Equivalence	4	45
Fraction Division	4	51
Concept of Unit	6	59
Ordering	10	61
Qualitative Questions	4	65
Part-Whole Concepts	12	69
Division Story Problems	5	78
Total Test Items	45	63

Test scores ranged from10 items correct to 44. The overal mean correct was 63 percent. While 10 students (20%) answered more than 84 % of the items correctly, another ten students answered less than 42% of the items correctly. Upon completion of the mathematics techniques class these twenty students will have completed all the required mathematics preparation needed to teach elementary school. But, unfortunately, there will be ten students who do not know (at least with respect to fractions) ideas that they will be certified to teach. It is unlikely that the mathematics techniques class they were enrolled in would have been sufficient to remediate their mathematics deficiencies.

Rational Number Test

The items selected for the test emphasized conceptual understanding of rational number ideas that should be part of the elementary school curriculum. Even though many of the items could have been solved procedurally, if the preservice teacher forgot the particular procedure, he/she could have relied on intuitive, less formal reasoning to solve the problem. For example, to determine the larger of this fraction pair, 16/31 or 9/19, changing both fractions to equivalent fractions with the same denominator is an unnecessary and tedious step. A decision can be reached by comparing to 1/2; 16/31 is more than 1/2 while 9/19 is less than 1/2. Results from the Rational Number Project (Post & Cramer, 1987) showed that fourth graders who participated in the rational number teaching experiment which emphasized the development of conceptual understanding over procedural skill did order fractions pairs as the one above by comparing the fractions to one-half.

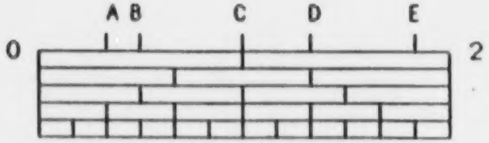
The fraction division problems were also solvable by relying on conceptual understanding of division. For example, 5/4 divided by 1/4 could be solved by thinking: *How many 1/4's in 5/4?* The least successful division problem (40% correct) was 1/2 divided by 4. While students struggled to find a procedure to solve the problem, they could have solved it by drawing a picture: 0________ı________1. If they divided the region from zero to 1/2 into 4 equal parts, each part would be 1/8. Students also could have thought of a number to multiply 4 by to obtain 1/2.

Examples of items from the different categores are described below. Percent correct for the particular sample item is also noted in the parentheses after the test question.

Part-Whole Concepts

Shade 3/8 of the circles (75%)

Here are 7 numberlines. All have a length of 2 units. The top line is labeled A, B, C, D, and E. In the space below write the numbers represented by each letter.



A (65%)
B (75%)
C (81%)
D (42%)
E (33%)

Ordering

Choose the larger fraction:

3/4 2/3 (83%)

.9 11/12 (60%)

16/31 9/19 (46%)

Name a proper fraction between 7/8 and 1 (54%)

This last problem shows students' reliance on procedures. The most common incorrect answer was the fraction 14/16 which, of course, was generated by multiplying the numerator and denominator by two. Other incorrect answers included, 8/8, 5/6, 4/8, 7/9.

Fraction Equivalence

3/8 = /12 (38%)

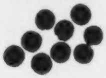
8/14 = / 21 (58%)

8/15 = /5 (27%)

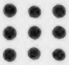
The lack of success with these items reflect the students discomfort with having *fractions within a fraction.* One student's comment that he thought you could not have fractions in the numerator most likely characterizes the students' misunderstanding. The students were more successful with the second problem because the fraction 8/14 could be reduced to 4/7. Now the problem 4/7 = /21 can be solved by multiplying numerator by the whole number 3 to obtain a whole number answer of 12.

Concept of Unit

These circles represent 3/7 of some unit. How many circles in the unit? (58%)



Below is 3/4 of some unit. Show 2/3. (56%)



••• ••• = 3/2. Show how many stars are there in the unit. (23%)

The concept of unit questions test a more indepth understanding of fraction concepts. To answer the first problem, for example, the student needs to reflect on what 3/7 means (1/7 + 1/7 + 1/7). By partitioning the picture into 3 equal groups the student sees that 1/7 equals 3 circles. Since 7/7 makes up the whole unit, 21 circles is the correct answer.

The third problem was more difficult because of the arrangement of the circles into two groups of three. Behr and Post (1981) called these problem types perceptual distracters and suggested that the ability to solve these problem types was one index of the stability of the learner's rational number concepts. If the student disregarded the way that the circles were clustered, the problem is essentially the same as the first problem (3/2 = 1/2 + 1/2 + 1/2 so 2 circles equals 1/2; now 2/2 = 4 circles). The two most common incorrect answers were 3 and 6.

Fraction Division

Simplify these complex fractions: $\frac{3}{4/3}$ (50%) $\frac{3}{1/3}$ (52%)

Though the fraction division problems were presented as complex fractions, most students knew to interpret them as division problems. Students struggled to find some procedure to solve the problems, but were inconsistent with the procedure they used. For example, some students who successfully answered the problems when the denominator was a fraction made an error on the problem where the denominator was a whole number. The division problem, (1/2)/ 4 , was changed to [1/2 x 4/1 = 4/2 = 2]. If the students reflected on the answer they would have seen the unreasonableness of it. A proper fraction divided by a whole number must result in an answer less than the fraction. The incorrect strategies applied to these questions yielded mostly unreasonable answers. The students did not attempt to assess the validity of their procedure based on the answers it generated.

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Fraction Story Problems

Mary has 1/2 yard of ribbon. She cuts it into 4 equal lengths. How long is each length? (73%)

The same numbers used in fraction division were embedded in fraction story problems. Interestingly, the most successful category was the fraction story problems (78% correct), while the second least successful category was fraction division (51%). Students had more understanding of fraction division than would have been attributed to them given the fraction division results. These preservice teachers had not thought to use their informal understanding of division to solve a problem presented in symbolic form but did so within a story context. No student actually wrote a division number sentence to solve the story problems. When given a problem in symbolic form, they only thought to solve it procedurally.

Qualitative Questions

What will happen to the fraction 3/15 if the numerator is increased while the denominator stays the same? (83%)

a) the value of the fraction decreases
b) the value of the fraction increases
c) the value of the fraction stays the same
d) there is not enough information to tell

Students were relatively successful on three of the four items in this category. When the question stated that the numerator and denominator were both increased, success level dropped to 40%. Since the answer depends on knowing if the increase is proportional or not, the correct answer was choice d, *there is not enough information to tell.*

Summary

Students were not as successful on this test as educators would expect them to be. As with the data from the Mathematics Leadership Project, there is a group of potential teachers (20% in this study) who do not have the mathematics background sufficient enough to teach fractions with understanding. As mathematics educators continue to suggest changes in the elementary curriculum, and to continue to emphasize teaching for conceptual understanding, we in the postion of training teachers need to be sure that new teachers have the needed mathematics understandings themselves to teach the mathematics we want them to teach in the way we want them to teach.

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PME-NA SYMPOSIUM

NATIONAL CENTER FOR RESEARCH IN MATHEMATICAL
SCIENCES EDUCATION:
CURRICULUM/ASSESSMENT WORKING GROUP

ORGANIZER: THOMAS A ROMBERG
DIRECTOR, NATIONAL CENTER FOR RESEARCH IN
MATHEMATICAL SCIENCES EDUCATION

The purpose of this symposium is threefold. First, we want to acquaint the members of PME-NA with the newly funded National Center and its mission. Second, we will outline the research agenda of the Curriculum/Assessment Working Group. And third, we want to encourage other researchers to collaborate with us in order to accomplish our mission.

The structure of the symposium and identification of discussants is as follows:

Overview of NCRMSE

- The Center's mission and the C/A research agenda.
  Tom Romberg

Curriculum Issues

- The potential influence of technology.
  Jim Kaput

- The potential impact of cognitive science and cultural sociology.
  Tom Romberg

Assessment Studies

- Teachers perceptions of the impact of testing on instruction.
  Steve Williams

- Mandated Testing in the U.S.
  Anne Zarinnia

Call for Collaboration

Tom Romberg

Magdalene Lampert
Institute for Research on Teaching
Michigan State University

THE TEACHER'S ROLE IN REINVENTING THE MEANING OF MATHEMATICAL KNOWING IN THE CLASSROOM[1]

ABSTRACT

This paper describes a research and development project in teaching designed to examine whether and how it might be possible to bring the meaning of knowing mathematics in school closer to what it means to know mathematics within the discipline by deliberately altering the roles and responsibilities of teacher and students in classroom discourse. The project was carried out as a regular feature of lessons in fifth grade mathematics in a public school. The teacher modeled a form of social interaction appropriate to making mathematical arguments in response to students' conjectures. Students were taught to examine the hypotheses about mathematical structures that underlie their solutions to problems. The outcomes of this teaching are described in terms of the abilities and propensities students develop to create mathematical arguments in a particular episode of classroom interaction.

A MATHEMATICAL PREFACE

In the midst of an argument among his students about a theorem in geometry, the teacher in Lakatos's *Proofs and Refutations* finds it appropriate to announce:

> I respect conscious guessing, because it comes from the best human qualities: courage and modesty (1976, p. 30).

Why does this teacher of mathematics think it appropriate to encourage conscious guessing and to celebrate the human virtues of courage and modesty? The answer to this question is to be found in Lakatos's analysis of what it means to know mathematics and his ideas about how new knowledge develops in the discipline.

In *Proofs and Refutations*, Lakatos portrays historical debates within mathematics about what the "proof" of a theorem represents by constructing a conversation among a group of students: fictional characters who voice the disagreements among mathematicians through the last several centuries, often using the mathematicians' own words. Lakatos's argument, which comes through in the person of the teacher, is that mathematics develops as a process of "conscious guessing" about relationships among quantities and shapes, with proof following a "zig-zag" path starting from conjectures and moving to the examination of premises through the use of counterexamples or "refutations." This activity of doing mathematics is different from what is recorded once it is done: "Naive conjecture and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end product." (Lakatos, 1976, p. 42). The product of mathematical activity might be justified with a deductive proof, but the product does not represent the process of coming to know. Nor is knowing final or certain, even with a proof, for the assumptions on which the proof is based-- which mathematicians call axioms-- continue to be open to reexamination in the mathematical community of discourse.[2] It is this vulnerability to reexamination that allows mathematics to grow and develop.

Mathematics has grown and changed over time, in Lakatos's view, not because the conclusions that are derived from axioms are the result of faulty logic, but because the axioms and definitions from which the logical argument begins are themselves open to revision as they are examined in the community of discourse. The need for revisions does not become obvious, however, until one engages in the process of proof, and discovers the shortcomings of one's assumptions. The insufficiency of the original assumptions come to be recognized as one tries to pursue their logical consequences rather than before the fact: refutations of the conclusions, often in the form of counterexamples, suggest revisions to the assumptions. Lakatos demonstrated that this zig-zag between revising conclusions and revising assumptions in the process of coming to know occurred both in the work of individual mathematicians as they exposed their work to their colleagues, and over time as conclusions that had been unquestioned in the past were reconsidered.

From the standpoint of the person doing mathematics, making a conjecture (or what Lakatos calls a "conscious guess") is taking a risk; it requires the admission that one's assumptions are open to revision, that one's insights may have been limited, that one's conclusions may have been inappropriate. While possibly garnering recognition for inventiveness, letting other interested persons in on one's conjectures increases personal vulnerability. Courage and modesty are appropriate to participation in mathematical activity because truth remains tentative, even as the proof of a conjecture evolves.

Polya also thought courage and modesty to be essential to the activity of acquiring mathematical knowledge. He asserted that the doer of mathematics must assume "the inductive attitude," and be willing to question both observations and generalizations, playing them off of one another in a form similar to what Lakatos called the zig-zag path from conjecture to proof and back to axioms.

> In our personal life we often cling to illusions. That is we do not dare to examine certain beliefs which could be easily contradicted by experience, because we are afraid of upsetting the emotional balance. There may be circumstances in which it is not unwise to cling to illusions, but [in doing mathematics]... we need to adopt the inductive attitude [which] requires a ready ascent from observations to generalizations, and a ready descent from the highest generalizations to the most concrete observations. It requires saying 'maybe' and 'perhaps' in a thousand different shades. It requires many other things, especially the following three:
>
> INTELLECTUAL COURAGE: we should be ready to revise any one of our beliefs.
>
> INTELLECTUAL HONESTY: we should change a belief when there is a good reason to change it...
>
> WISE RESTRAINT we should not change a belief wantonly, without some good reason, without serious examination.
> (1954, pp. 7-8)

Polya called these the "moral qualities" required to do mathematics. He recognized that examining one's assumptions is an emotionally risky matter, but like Lakatos, he claimed that it was essential to doing good mathematics.

COMING TO KNOW MATHEMATICS IN SCHOOL

These views of what doing and knowing mathematics entail are radically different from those that are assumed in popular culture, and in most classrooms.[3] Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly (Schoenfeld, 1985; Stodolsky, 1985; Ball, 1988). These cultural assumptions are shaped by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher. These beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing.

Comparing school mathematics with knowing in the discipline

The issue of intellectual authority is central to this comparison between how mathematics is known in school and how it is known in the discipline. In the classroom, the teacher and the textbook are the authorities and mathematics is not a subject to be created or explored. In school, the truth is given in the teacher's explanations and the answer book; there is no zig-zag between conjectures and arguments for their validity, and one could hardly imagine hearing the words "maybe" or "perhaps" in a lesson. Knowing mathematics in school therefore comes to mean having a set of unexamined beliefs, whereas Lakatos and Polya suggest that the knower of mathematics needs to be able to stand back from his or her own knowledge, evaluate its antecedent assumptions, argue about the foundations of its legitimacy, and be willing to have others do the same. Although the teacher tells students whether their answers are right or wrong, it seems that few teachers engage students in a public analysis of the assumptions that they make to get their answers. Even when teachers give an explanation rather than simply stating a rule to be followed, they do not invite students to examine the mathematical assumptions behind the explanation, and it is unlikely that they do so themselves (Ball, in press). In conventional mathematics lessons, students believe that the teacher knows which answers are right, and teachers believe that the paths to these answers can be found in rules in books; examining these beliefs would, in Polya's words, "upset the emotional balance" in both teachers and students (Cooney, 1987). That teachers and students think this way about mathematical knowledge and how it is acquired is both a cause and a logical consequent of the ways in which knowledge is regarded in school mathematics lessons.

Must there be this disjunction?

This paper is a description of a research and development project in mathematics teaching designed to explore whether and how it might be possible to produce lessons in which public school students would exhibit-- in the classroom-- the qualities of mind and morality that Lakatos and Polya associate with doing mathematics.[4] My role in this project has been to develop and implement new forms of teacher-student interaction as well as to experiment with new forms of content as a teacher of fifth grade mathematics. I will present some of the findings of this research and development project here in terms of a story about learning about knowing mathematics in the social setting of the classroom. I will tell the end of the story first by describing the activity of a class of fifth graders who seem to have learned to do mathematics together in a way that is consonant with Lakatos' and Polya's assertions about what doing and knowing mathematics entails. The evidence for their learning will be presented in the form of an exhibit of what they are able and inclined to do with their teacher and with one another in the social context of a lesson. The students are courageous and modest in making and evaluating their own assertions and those of others, and in arguing about what is mathematically true; they move around in their thinking from observations to generalizations, and back to observations to refute their own ideas and those of their classmates. While they are learning mathematics, these students are also learning, tacitly if not explicitly, to place mathematics appropriately in the lexicon of ways of knowing. Their activity suggests that they are operating with quite a different set of beliefs about what doing mathematics means than those that are held by other fifth graders in similar school settings (Stodolsky, 1988). In acting on what they believe about the relationship between the knower and what is known, they put themselves in the position of authors of ideas and arguments; in their talk about mathematics, reasoning and mathematical argument-- not the teacher or the textbook-- are the primary source of an idea's legitimacy.

How does learning what it means to know mathematics come about?

In addition to illustrating what the students have learned about how to participate in the doing and learning of mathematics, the episode will also show how the teacher acts to create and maintain the culture in which such student activity can occur. This particular episode was chosen to represent the mathematics lessons I have designed and taught because it illustrates several patterns of teacher and student interaction that can be shown to be common to almost every lesson over the entire school year. Research suggests that such patterns can be initiated by the teacher to build a "participation structure" that redefines the roles and responsibilities of both teacher and students in relation to learning and knowing (Au & Jordan, 1980; Au & Mason, 1981). The notion of a classroom participation structure is taken from the sociolinguistic work of Florio (1978) and Erickson and Shultz (1977). They define a participation structure to be the allocation of interactional rights and obligations among participants in a social event; it represents the consensual expectations of the participants about what they are supposed to be doing together, their relative rights and duties in accomplishing tasks, and the range of behaviors appropriate within the event.[5] Teachers and students thus form communities of discourse who come to agree on working definitions of what counts as knowledge and the processes whereby knowledge is assumed to be acquired (Cazden, 1988).

Shared meaning

In the classroom, words like "know", "think", "revise", "explain", "problem", and "answer" come to have meaning by being associated with particular kinds of activities. Who is responsible for doing the activities associated with these words gets determined in interaction between the teacher and the students. In conventional lessons the participants agree that it is the teacher's responsibility to explain and the students' responsibility to give answers. Problems are questions, and finding the correct answers to those questions is an indication of knowing mathematics. The teacher is responsible for knowing is whether the answers a student gives are correct and to be asked by the teacher to revise means "you've done it wrong" and "you've got to do it over" until more of the answers are correct. When the teacher asks a students to think it is often an admonition to be quiet; thinking is not considered to be a process that students-- or teacher and students-- engage in together.

What is the teacher's role?

Teaching is not only about teaching what is conventionally called "content". It is also teaching students what a lesson is and how to participate in it (Jackson, 1968; Florio, 1978; Mehan, 1979). From the activities the teacher sets for them, students learn what counts as knowledge and what kind of activities constitute legitimate academic tasks

(Lemke, 1982; Doyle, 1985, 1986; Leinhardt & Putnam, 1988; Cazden, 1988). Face to face interaction between students and their teacher follows context specific rules, and cues within these contexts signal how what anyone says is to be understood in relation to the task everyone is assembled to accomplish (Mehan, 1979; Cazden, 1988). The teacher has more power over how acts and utterances get interpreted, being in a position of social and intellectual authority, but these interpretations are finally the result of negotiation with students about how activity is to be regarded.

In order to challenge conventional assumptions about what it means to know mathematics, then, teachers and students need to do different sorts of activities together, with different kinds of roles and responsibilities.

In my interactions with students, I associated mathematically appropriate activities with words like "know", "think", "revise", "explain", "problem", and "answer" to initiate a redefinition of these activities, and I demonstrated to students what new roles and responsibilities-- for them and for me-- the new definitions entailed.

WHEN THE PROBLEM IS NOT THE QUESTION AND THE SOLUTION IS NOT THE ANSWER: INVENTING NEW FORMS OF TEACHER-STUDENT INTERACTION

How does a teacher[6] go about redefining the meaning of "knowing mathematics" ? In the lessons I designed and enacted, I portrayed what I wanted students to learn about mathematical knowing both in how I constructed my own role and in what I expected of the class. I gave them problems to do, but I did not explain how to get the answers, and the questions I expected them to answer went beyond simply determining whether they could get the solutions. I also expected them to answer questions about mathematical assumptions and the legitimacy of their strategies.

Answers to problems were given by students, but I did not interpret them to be the primary indication of whether they knew mathematics. In this interaction, the words "knowing", "revising", "thinking", "explaining", "problem", and "answer" took on new meanings.

Students' knowing as conjecture and proof

In setting the topical agenda in any particular lesson--that is, in determining what the mathematical discourse was going to be about--my responsibility was to choose a problem or problems for discussion and the students' responsibility was to express their interests, questions, and understandings within the domain of the problem. As it was realized during a discussion the agenda evolved from a negotiation between the problems I set and the mathematics that students brought to working on those problems (Cf. Steffe, 1988). The problems communicated predictable boundaries for the class discussion, enabling students to know what they are supposed to be doing and thinking about during the class period (Cf. Leinhardt and Putnam, 1987).

At the beginning of a unit, when we were switching to a new topic, the problem we started with was chosen for its potential to expose a wide range of students' thinking about a bit of mathematics, to make explicit and public what they could do, and how they understand. Later problems were chosen based on an assessment of the results of the first and subsequent discussions of a topic, moving the agenda along into novel but related mathematical territory. The most important criteria in picking a problem was that it be the sort of problem that would have the capacity to engage all of the students in the class in making and testing mathematical hypotheses. These hypotheses are embedded in the answers students give to the problem, and so comparing answers engaged the class a discussion of the relative mathematical merits of various hypotheses, setting the stage for the kind of zig-zag between inductive observation and deductive generalization that Lakatos and Polya see as characteristic of mathematical activity.

For example, in the class described below, students asserted various hypotheses about how to figure out the last digit in 5^4, 6^4, and 7^4 without multiplying.[7] In order to push them to speculate on whether their hypotheses would hold in a larger domain, I then asked them to think about what the last digit would be in 7^5. Two competing hypotheses about how exponents work were revealed in their assertions about the last digit in 7^5. One student's initial hypothesis was that the fifth power would be obtained by squaring the fourth power. Other students hypothesized that the fifth power would be obtained by multiplying the fourth power by the base number (or first power). In arguing about whether the last digit in 7^5 was "1" or "7", the students were arguing about which of these hypotheses was an appropriate "law" to use in their further work with exponents.

Problems like these are what Kilpatrick, following Frederickson, calls "structured problems requiring productive thinking" (1987, p. 134). They imply criteria for testing the correctness of the solution, but they are not solved by the simple application of a known algorithm. This allows for multiple routes to a solution, and puts the solver in the position of devising all or part of the solution procedure. The students' responsibility is figuring out how to solve the problem as well as finding the solution. It is the strategies used for figuring out, rather than the answers, that are the site of the mathematical argument, and it is these strategies reveal the assumptions a student is making about how mathematics works. It is in this sense that "the problem is not the question and the answer is not the solution". This approach to teacher and students talking about the problem and its solutions is intended to communicate that what is important is developing and defending strategies, making hypotheses, or what Lakatos calls "conscious guessing," and rising to the challenge of articulating and defending the assumptions that led up to your guess.

How do the answers to problems work to express mathematical knowing?

As students volunteered their solutions to a given problem, I wrote them all up on the board for consideration, and I put a question mark next to all of them. Often I put students' names next to their answers, as a help for everyone to remember who they are to address if they have something to say about that answer. The names also are meant to indicate that the answers still belong to the persons who figured them out, even though they have been given to the teacher. When someone asserted that one of the answers should be eliminated because it was incorrect, it was considered fair game for the teacher to ask anyone in the class to explain why.

Once the list of students' solutions was up on the board, they were open for discussion and revision. Students often began by explaining why they gave the answer that they did. If they wanted to disagree with an answer that was up on the board, the language that I have taught them to use is, "I want to question so-and-so's hypothesis." (Until the group arrived at a mutually agreed upon proof that one or more of the answers must be correct, all answers were considered to be "hypotheses".) I always asked them to give reasons why they questioned the hypothesis so that their challenge takes the form of a logical refutation rather than a judgement. The person who gave the answer was free to respond or not with a revision. In order to communicate the idea that I thought every answer was (or should be) arrived at by a process of reasoning that makes sense to the person who volunteered it, I asked the class, "Can anyone explain what they thought so-and-so was thinking?" and "Why would it make sense to think that?" And then I asked the person who gave the answer to respond. This routine was a way of modeling talk about thinking. It also made thinking into a public and collaborative activity, wherein students would rehearse the sort of intellectual courage, intellectual honesty, and wise restraint that Polya considered to be essential to doing mathematics.

The teacher as a representation of what it means to know mathematics

Simply by virtue of having had more education, the teacher represents the most expert knower of mathematics in the classroom, and in this role, has the potential to demonstrate the nature of expertise to those who seek to acquire it. Given my goal of teaching students a new way of knowing mathematics, I needed to demonstrate what it would look like for someone more expert than they to know mathematics in the way I wanted them to know it. The role I took in classroom discourse, therefore, was to follow and engage in mathematical arguments with students; this meant that I needed to know more than the answer or the rule for how to find it and I needed to do something other than explain to them why the rules worked: I needed to know how to prove it to them, in the mathematical sense, and I needed to be able to evaluate their proofs of their own mathematical assertions.[8]

For students to see what sort of knowing mathematics involves, the teacher must make explicit the knowledge she is using to carry on an argument with them about the legitimacy or usefulness of a solution strategy. She needs to follow students' arguments as they wander around in various mathematical terrain and muster evidence as appropriate to support or challenge their assertions, and then support students as they attempt to do the same thing with one another's assertions. As the teacher moves around in mathematical territory in a flexible manner, she is modeling an approach to problem solving. She is demonstrating what it means "to know" what the mathematician Henry Pollak calls "cross-country" mathematics. In contrast to walking on a well-marked path, the cross-country terrain is jagged and uncertain; watching someone traverse it is a key to learning how to traverse it yourself. Pollack said of his teacher, Ed Begle:

> [As a student, I] had a very interesting time watching him struggle, inventing proofs and trying to think about the right way to do it. I learned a lot more mathematics that way than I might have if it had been a perfectly polished lecture; and I think already at that time I developed my feeling that I like cross-country mathematics. Mathematics, as we teach it, is too often like walking on a path that is carefully laid out through the woods; it never comes up against any cliffs or thickets; it is all nice and easy (Albers & Alexanderson, 1985, p. 231).

If the teacher only demonstrates that she knows how to explain the rules, and whether or not students' answers are correct, the student will get an unfortunately limited picture of mathematical expertise, and it is unlikely that he or she would learn how to walk on any but the most well-marked paths.

Summary

These patterns of social interaction were designed to involve me, as the teacher, in three different kinds of teaching about what it means to know mathematics. Sometimes, I straightforwardly told students what kinds of activities were and were not appropriate. At other times, I modeled the roles that I wanted them to be able to take in relation to themselves and one another. At at other times, I did mathematics with them, just as a dance instructor dances with a learner so that the learner will know what it feels like to be interacting with someone who knows how to do what he or she is trying to learn how to do.[9] Just as the dance instructor knows traditional forms of dance, and demonstrates them, I demonstrated the conventions of mathematical discourse to my students, but we also reinvented them as we did mathematics together. Students also assumed the role of more experienced knowers in relation to one another as they became more competent in the interactional routines associated with mathematical discourse in the classroom. The lesson described below represents a moment in this process, when I would claim that the students had learned to regard themselves as a mathematical community of discourse, capable of ascertaining the legitimacy of any member's assertions using mathematical form of argument.

AN EPISODE OF TEACHING AND LEARNING ABOUT KNOWING MATHEMATICS

Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of determining what is true. Students learn about how the truth of a mathematical assertion gets established in mathematical discourse as they zig-zag between their own observations and generalizations-- their own proofs and refutations-- revealing and testing their own definitions and assumptions as they go along.

In the lesson I will describe here, the mathematical content of the discourse was the operation of exponentiation: writing numbers as "powers" of other numbers, and comparing their orders of magnitude. Exponentiation is a powerful mathematical tool, used both in the development of new theory and in designing applications of mathematics to the solution of scientific problems. The power of exponents lies in the idea that you can compare numbers by comparing their orders of magnitude, thereby simplifying the mental operations involved in the comparison; one technical implication of that simplification is that you can multiply large numbers by adding exponents, and divide by subtracting them. If you wanted to compare the populations of two cities, for example, one of which was given as 42,123,184 persons and the other of which was given as 421,032 persons, you could divide one number into the other. Even using a calculator, however, this is a tedious and messy procedure, and it does not make use of the best mathematical tool available to do the job. It is more intellectually economical to think, not of the particular digits in the numbers, but of the first number as approximately 4 x 10,000,000 and the second as approximately 4 x 100,000. If we write these numbers as 4×10^7 and 4×10^5, we have an efficient tool for finding out how many time the population of the smaller city would fit into the larger. Operating on these numbers using the laws of exponents involves subtracting five from seven. The difference is two, and that means that the larger city is 10^2 or 100 times the population of the smaller city. Although this way of thinking about the relationship between the two numbers is more complicated to explain than division, once you know how to use exponents to think about numbers, the comparison process becomes much simpler. But how do we kno for sure that it is legitimate to add and subtract exponents as a way of multiplying and dividing? Why is it true that 4×10^7 divided by 4 x

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10^5 is 10^2? In mathematics we can prove that this is true, given certain assumptions about the meaning of exponents, by the process of mathematical argument: the process of observation and generalization that Lakatos and Polya describe.

I wanted my students to learn not only that they could divide or multiply by subtracting or adding exponents, or how to use the technology of exponents, but also that the warrant for doing so comes from mathematical argument and not from a teacher or a book. This meant that I needed to work on two teaching agendas simultaneously. One agenda was related to the goal of students' acquiring technical skills and knowledge in the discipline which could be called knowledge of mathematics, or mathematical content. The other agenda, of course, is working toward the goal of students' acquiring the skills and dispositions necessary to participate in disciplinary discourse, which could be called knowledge about mathematics, or mathematical practice (Schoenfeld 1985; Wilson, 1988; Ball, 1988). These two kinds of knowledge interact: in learning about discourse, students learn about what kind of knowledge they have when they know how to use exponents. At the same time, knowledge of tools, vocabulary, and symbols provide students with the "cognitive technologies" that enable them to make arguments of a substantially different sort than they would be able to make without them (Pea, 1987; Vygotsky, 1978). The interaction between learning to use the tools that have become a part of mathematical culture and inventing new mathematics by reasoning may be thought of as the process by which the individuals in the community of discourse that is the school class come to know mathematics. As such, it parallels the development of knowledge in the discipline.[10]

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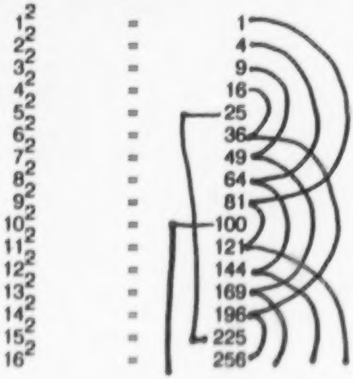
The content of the lesson as an expression of the nature of mathematical knowledge

The nature of mathematical knowing is such that you can find out something about the characteristics of unknown quantities by studying patterns in numbers that you can observe (Polya, 1954). This is the essence of the connection between learning about exponents and learning about mathematical knowledge. I started the unit on exponents by asking all of the students in the class to prepare their own tables of squares from 1^2 to 100^2 using calculators. I challenged them to find patterns in the tables, and they were actively engaged in doing so for at least three forty-five minute periods. During this activity, they invented a way of thinking about relationships among the numbers upon which I could build to take them into new mathematical territory: they developed the idea of talking about patterns in powers by focusing on the "last digits" of numbers, rather than thinking about all of the digits in a number at once. This was a tool, developed within the mathematical community of discourse in my classroom, that could be used to move away from the particular numbers that were represented in the list of square numbers, toward the characteristics of those numbers and the structure of relationships among them. The connection between learning about exponents as mathematical content and learning about the nature of mathematical knowledge by engaging in discourse about exponents is that in mathematics, you can find out something about the character of unknown numbers by studying patterns in numbers that you know. For my class of fifth graders, looking for patterns in the "last digits" of powers was a way into knowing about what sort of predictive power can be derived from the analysis of quantitative order.

Looking at last digits is a mathematically appropriate cognitive tool; it reduces the amount of information you need to pay attention to in searching for patterns. The first assertion that students in my class made about patterns in square numbers was that the last digits of the squares alternated between being odd and even, just as the base numbers did. Their most sophisticated conjecture about squares was that: the square of a multiple of ten would always be a multiple of ten and so end in a zero, the square of a number ending in five would also always end in a five, and in between the zeroes and fives, the strings of last digits would always be symmetrical around both zero and five. They used physical representations like this to work out this conjecture:



$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$10^2 = 100$
$11^2 = 121$
$12^2 = 144$
$13^2 = 169$
$14^2 = 196$
$15^2 = 225$
$16^2 = 256$

and so on...

and they proved that the pattern would continue "forever" by arguing that numbers that ended in one or nine would have to have squares that ended in one, numbers that ended in two or eight would always have squares that ended in four, numbers that ended in three or seven would always have squares that ended in nine, and numbers that ended in four or six would always have squares that ended in six. Since the last digits of whole numbers are arranged in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, and this arrangement repeats itself over and over again, the square numbers would follow the doubly symmetrical pattern as long as there were base numbers to associate them with. It was at this point in the unit on exponents that the lesson I want to describe here occurred. I used their idea of looking at what happens in last digits to elicit some conjectures about how to operate with exponents.

The lesson began when I wrote on the blackboard at the beginning of class:

What is the last digit in: 5^4?
6^4?
7^4?

and I challenged the class to tell me if they could prove that their conjectures about what these last digits would be were true without doing the full multiplications. I walked around the classroom, watching and listening to what they are doing, and when everyone seemed to be engaged in the task, and most had given it some thought, I began a whole class discussion. That discussion had three parts: a clarification of terms, symbols, and definitions; a consideration of the special properties of powers of five and six; and speculations about powers of seven that led to more general hypotheses about how exponents could be operated on to do arithmetic operations more efficiently.

Agreeing on what exponents mean

In this part of the discussion, teacher and students were finding out what other members of the group understood about the operations indicated by exponents, and coming to some agreement about common language to use in their conversation. Students started doing some "conscious guessing" about

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how exponents work as they were defining the terms of the argument. The first part of the class session focused on the fourth power of five. I began by saying, "Who is ready with a theory about this one?" pointing to 5^4 on the board. Narinia asserted that, "It's going to be the same as the squares. First you take five and square it, and you get twenty-five, and the last digit of that is five, and you square that." Alianna then said, "Yeah. You square the five two times." Her tone of voice suggested agreement with Narinia, but what she said would not unambiguously describe the procedure that Narinia had followed. In previous classes, Alianna and other students had interpreted expressions written as "x^n" to mean "multiply x by n." We had had some discussion then about why exponential notation might lead to this confusion.

I wrote on the board:

> Alianna: Square five two times.
> Narinia: Square five and then square that.

I said, "I think we have a language problem here. Do those two directions mean the same thing?" Martha responded saying, "They might think you have to square five and times it by two." And I wrote "25 x 2" on the board again, saying, "Do you mean this?" At this point, Alianna came back into the conversation and said, "No, cause it's square it, again. Also, if you said 'Square five two times, they might add it and get fifty [25 + 25]." So I asked, "How should you say it?" Narinia reentered the discussion, clarifying what she had said earlier: "Square five, and then square the answer to that." Carl also had his hand raised, and said, "Square the squared number of five." And Gar piped in, "I was going to say what Narinia said." Martha then thought it appropriate to make a statement about how she interpreted the conversation: "I understood both of them. I did five times five and I did five times five again, but I didn't add them, I multiplied them."

Discussion[11]

There are several mathematically legitimate ways to figure out the last digit in 5^4. One could multiply 5 X 5 and then multiply the answer to that by five, and multiply the answer to that by five: 5 x 5, then 5 x 25, then 5 x 125, to get 625. What the students were trying to figure out-- the problem they were solving-- was whether there was a more efficient way to do it that was also mathematically defensible. As they argued about this, they clarified their language and use of symbols. They and the teacher need to be able to have a conversation using terms that are functional, not only for communication but for reasoning. As a representative of mathematical culture outside of the classroom, the teacher brings conventional mathematical tools (including language and symbols) into the discussion and negotiates their meaning with students to add to the tools they are able to use to enhance their thinking. In the first part of the class this entailed the teacher checking in with those students who had in past discussions interpreted "the power of two" to mean "multiply by two." Although exponentiation entails repeated multiplication, just as multiplication entails repeated addition, it is a qualitatively different operation from multiplication and it works by different rules (Asimov, 1961). Unless this difference between multiplying by two and the power of two was cleared up, these students would be unable to productively participate in arguments about how the fourth power of a number might be related to the second power, and go on to assert appropriate rules for working with powers.

Conscious guessing, and proof that zig-zags between induction and deduction

After this discussion of how we would talk about the operations ndicated by exponents, Sam asserted, about the last digit in 5^4: "It has to end in a five." I invited everyone in the class to consider the validity of Sam's decisive assertion and to see if they could explain why he seemed to be so sure. The question I was asking was: How does he know that is true? Harriet said, "Well, anything multiplied by five has to end in a five or a zero," and Theresa quickly added, "but it has to be a five because when you multiply five times five you get a five [for a last digit]." Martha observed, "You times the square number, you square it again and you get six hundred and twenty-five." And Karl responded, moving to the level of a mathematical generalization, "You don't have to do that. It's easy, the last digit is always going to be five because you are always multiplying last digits of five, and five times five ends in a five." Karl went beyond the question about the last digit in 5^4 and gave both a conjecture and a proof about what must be true of the last digit of all of the powers of five.[12]

At this point, some other students, who had been working at finding higher and higher powers of five with their calculators, made the conjecture that all powers of five would end in "25" for their last two digits, and they made some arguments for why that might "always" be true, no matter how high the power. In doing so, they also zig-zagged between observations and generalizations. Interestingly, their deductive arguments relied on the framework of the conventional multiplication algorithm, even though they had been using the calculator to develop their hypothesis inductively: they talked about what would "have to happen" in the "ones" and the "tens" columns every time you multiplied and carried in successive multiplications of a number that ended in "25" by five.

We had a short discussion of the powers of six, which seemed to be of little interest because "they worked just like fives" (i.e. everytime you multiply a number that ends in six by another number that ends in six, the result will end in six). There was a conjecture that this similarity in last digits might be true for the powers of every number from one to nine, but someone quickly refuted it with a counterexample: 7^2 is 49; the power does not end in the same digit as the base number. Some students were talking among themselves about the fact that the last two digits in the powers of six were not always "36," which they had expected, based on an analogy to five where the last two digits were always "25".

Discussion

In this part of the lesson, students had begun to make assertions that were based on their inductive observations of patterns, and to move back and forth between these observations and deductive arguments about why the patterns would continue, even beyond the numbers they had tested. They ranged beyond speculating about the fourth power, which was the focus of the initial problems, and made conjectures about all of the powers of five and six. They began by stating reasons why anyone might "know for sure" that the last digit of a power of five would always be "5", and then went on to explore the boundaries of the theory that "If the last digit is always n when you raise n to some power, the last two digits will always be n^2." This theory worked for $n = 5$; they proved that the last two digits of any power of five would be "25" (or 5^2). Because the powers of six always ended in a six, they wondered if the two-digit conjecture would also work for $n = 6$. Two students asserted that (i.e. saying their idea as collaboration: "Me and Hudson think that...) the powers of six should end in "36", but someone else quickly found a counterexample.

In the course of this discussion, students shifted around from talking about "What I did" to figure out the problem to "What you do," distancing themselves from the procedures they were evaluating and making their assertions more general. In this process, their beliefs became justified and they moved into the realm of mathematical truths that were legitimated by the community of discourse.

There is evidence in this segment of the lesson that students appreciate that mathematics is about finding efficient strategies to solve problems; in addition to the student-initiated search for two-digit patterns, this showed up when Karl made a point of saying the "You don't have to" figure out that 5^4 is "625" in order to know that its last digit must be five, and then went on to support his argument with mathematical evidence. He was calling the class's attention to what doing mathematics means, and giving a particular example of what is entailed in doing it. It is not that multiplying 5 x 5 x 5 x 5 does not prove that the last digit in 5^4 is 5, but this proof is particular, not general; it is not as "elegant" as one that focuses only on what must be true about the last digit that results from each step in the operation.

When the answer is not the end of the discussion

When I asked if anyone had any ideas about 7^4, there was quick agreement that it should be "one" and I asked for proof. Gar said: "Seven times seven is forty-nine. And nine times nine is eighty-one. So take the one." Embedded in this argument is the assumption that $7^4 = 7^2 \times 7^2$ and that all you need to attend to is the last digit all the way through the procedure. Gar multiplied 9 x 9 to arrive at "1"; he did not multiply 49 x 49, and then try to figure out the answer to that. Versions of his argument were given and elaborated by other students. There was a repetition in much of what they said of the language that had been used in discussing 5^4, but the same terms and sentence structures were being used by different students. By this point, nearly everyone had had something to say.

Then I asked, to further elicit their conjectures about how exponents work: "What about seven to the fifth power?" Several students raised their hands after a few moments, and the ones that were called on said in quick succession:

Arthur: I think it's going to be a 1 again.
Sarah: I think it's 9.
Soo Wo: I think it's going to be 7.
Sam: It is a 7.

I wrote on the board:

$$\begin{aligned} 7^5 &= 1? \\ & 9? \\ & 7? \end{aligned}$$

and said: "You must have a proof in mind, Sam, to be so sure." and then I asked, "Arthur, why do you think it's one?"

The following discussion ensued as the students attempted to resolve the problem of having more than one conjecture about what the last digit in seven to the fifth power might be. It was a zig-zag between proofs that the last digit must be "7" and refutations of Arthur's and Sarah's alternative conjectures. ranged between observations of particular answers and generalizations about how exponents-- and numbers more generally-- work. Students examined their own assumptions and those of their classmates. I assumed the role of manager of the discussion, and sometimes participated in the argument, refuting a student's assertion.

Teacher: Arthur, why do you think it's one?
Arthur: Because 7^4 ends in 1, then it's times 1 again.
Gar: The answer to 7^4 is 2401. You multiply that by 7 to get the answer, so it's 7 x 1.
Teacher: Why 9, Sarah?
Theresa: I think Sarah thought the number should be 49.
Gar: Maybe they think it goes 9, 1, 9, 1, 9, 1.
Molly: I know it's 7, 'cause 7 . . .
Abdul: Because 7^4 ends in 1, so if you times it by 7, it'll end in 7.
Martha: I think it's 7. No, I think it's 8.
Sam: I don't think it's 8 because, it's odd number times odd number and that's always an odd number.
Carl: It's 7 because it's like saying 49 x 49 x 7.
Arthur: I still think it's 1 because you do 7 x 7 to get 49 and then for 7^4 you do 49 x 49 and for 7^5, I think you'll do 7^4 times itself and that will end in one.
Teacher: What's 49^2?
Soo Wo: 2401.
Teacher: Arthur's theory is that 7^5 should be 2401 x 2401 and since there's a 1 here and a 1 here . . .
Soo Wo: It's 2401 x 7.
Gar: I have a proof that it won't be a 9. It can't be 9, 1, 9, 1, because 7^3 ends in a 3.
Martha: I think it goes 1, 7, 9, 1, 7, 9, 1, 7, 9.
Teacher: What about 7^3 ending in 3? The last number ends in ... 9 x 7 is 63.
Martha: Oh . . .
Karl: Abdul's thing isn't wrong, 'cause it works. He said times the last digit by 7 and the last digit is 9, so the last one will be 3. It's 1, 7, 9, 3, 1, 7, 9, 3.
Arthur: I want to revise my thinking. It would be 7 x 7 x 7 x 7 x 7. I was thinking it would be 7 x 7 x 7 x 7 x 7 x 7 x 7 x 7.

Once Arthur revised his idea that the last digit should be "1" there were no further disagreements in the class with the conclusion that it should be "7". There was little time left, and I used it to extend the hypotheses students had been developing about how exponents work into a different domain.

Teacher: What power's that? (i.e. 7 x 7 x 7 x 7 x 7 x 7 x 7 x 7)
Arthur: 8th.
Teacher: That's 7^4 squared.
On board: $7^8 = 7^4 \times 7^4$
Teacher: What's 7^{16}, Arthur?
Arthur: It's going to be 7^8 x 7^8.

Julio: I think $7^{16} \times 7^{16}$ is going to be 7^{32}. It just doubles.
Soo Wo: Since 7^3 is 343, I think 7^6 would be $7^3 \times 7^3$, which would be 343 x 343.
On Board: $7^4 = 7^2 \times 7^2$
Teacher: Time is up. We have to stop. We'll continue on Monday.

At this point, some students were verging on declaring the second law of exponents: $(n^a)(n^b) = n^{a+b}$, which they would articulate more fully, and prove the legitimacy of, in the next few classes. They were also beginning to develop a modular arithmetic of "last digits" to go with different base numbers, leading them into further generalizations about the properties of exponents.

Discussion:

By the end of the lesson, fourteen of the eighteen students present in the class had had something mathematically substantial to say about exponents: an interpretation of language or symbols, an assertion about a pattern, a proof that a pattern would continue beyond the observed data, or an interpretation of another student's assertion. Each of these kinds of contributions is both an expression of what the student knows and a rehearsal of what the student believes to be an appropriate contribution to a school lesson in mathematics.

This final segment of the lesson represented a shift, from the problems that were set at the beginning of class to a new kind of problem. It was an extension of the domain of discourse from patterns that would apply to numbers raised to the fourth power, to strategies for figuring out what would be the last digit in a number raised to the fifth power. In the case of five and six as a base number, the question was trivial, but in the case of seven, it was not. The problem thus provides a forum in which assumptions that might have been made about how exponents work in a special domain could be tested on a new population with different characteristics. From the teacher's point of view, this shift has at least two functions. It extends the content of the lesson into new territory, and it serves to assess students' "conceptual competence" (Greeno, Riley, and Gelman, 1984) by assessing whether they could bring a conceptual structure developed in a more familiar domain to bear on one that was less familiar and more complex.

I purposely did not ratify any of the students' assertions about the answer to 7^5, nor their arguments for their various positions. When Arthur says, "I want to revise my thinking," he is using a phrase that he and the rest of the class have been taught to use and encouraged to use when they want to change their minds about an assertion made earlier in the discussion. It carries a different message than saying "My answer was wrong, and now I know the right one." When a student is in charge of revising his or her own thinking, and expected to do it publicly, the authority for determining what is valid knowledge is shifted from the teacher to the student and the community in which the revision is asserted.

Arthur maintained his intellectual courage through the course of this discussion, not changing his belief without revising the assumptions on which it was built. He was not inclined to revise by the repeated assertions of other students that the last digit was seven because he had made the guess that 7^5 would be obtained by multiplying 7^4 by 7^4, perhaps because we had established that $7^4 = 7^2 \times 7^2$. He was searching for a mathematical law, rather than just looking for the answer, and his conjecture was that the next highest power would be obtained by squaring the power below it. He revised his conjecture to assert that squaring a power would result in a power that was double the one you started with, so that 7^{16} could be obtained from the product of 7^8 and 7^8, 7^{32} could be obtained from the product of 7^{16} and 7^{16}, and so on. Beginning with a guess and exploring it with courage and modesty, he arrived at a bit of mathematical truth. What he was learning was both the laws of exponents, and how to justify that they work within the domain of mathematics.

Gar also exhibited intellectual modesty in his argument against Sarah's assertion that 7^4 would end in a nine. He worked at figuring out why she might think that, and then explained the logical contradiction in her assumption. He did not tell Sarah she was wrong. He left that judgement to her. Sarah never does get to tell the class why she thought the last digit should be nine, and there is no verbal record of whether she revised her thinking. (One would need to explore Sarah's thinking about exponents on some later occasion to find out more about whether her ideas about how they worked were congruent with mathematical conventions.) The same could be said of the interaction between Martha and Sam over whether the last digit could be "8".

The role that the I took as the teacher, in relation to Arthur's assertion that the last digit in 7^5 should be "1", was to ask him to explain, to monitor the tone of other students who wanted to disagree, to assert what some of Arthur's assumptions might have been, and to ask Arthur to articulate the revision that he had made. Throughout this part of the class, the teacher acted to support the idea that "Arthur's thinking" should be everyone's concern. The lesson was not just getting Arthur to rethink his assumptions, but helping everyone to see why those assumptions had led him to the conclusion that 7^5 would end in a "1". Both students and teacher have a different relation to the subject matter in this kind of discourse than they would in a conventional "knowledge telling" exchange. Arguments for the conclusion that the last digit in 7^5 should be "7" were given by other students repeatedly throughout the discussion. These arguments could be interpreted as "knowledge telling" by students who wanted the teacher to know what they were thinking, but they had the dual purpose of attempting to convince other members of the class that this conclusion was valid. Since they were asserted interactively with other assertions, there is some evidence that they may have been intended to have this argumentative purpose. On many occasions when such discussions occurred, one student might give several different arguments for why his or her conclusion was valid, and sometimes students would even ask to come up to the board to illustrate an argument for the class.

Worth noting in this part of the discussion is how often (15 times in about 5 minutes) and how many (8 out of 20) students say "I think" or "I don't think" and then make an assertion. An analysis of their assertions shows that they are no mere expressions of opinion; they recount their own reasoning processes and analyze those of others. They use "I think" to mean "I have figured out that" and their assertions of what they have figured out are regularly followed by arguments for why their strategies seem valid. They are indicating authorship of the ideas that they assert, and they are also indicating that thinking is something that a student both does and talks about in a mathematics class. Making assertions in this form is an expression of what they believe about roles and responsibilities in relation to mathematical knowledge, and where they put themselves in relation to the establishment of valid arguments in the discipline.

This use of language also serves to signify where the student is in the mathematical journey from a conscious guess to a proven theorem. Students who are sure that they can support their assertions say "It is" or "It has to be" or "I know." Saying "I think" rather than "It is" protects the student from associating his or her sense of self with an assertion that is later revised because it has been proven wrong.[13] This level of engagement with mathematical ideas may be similar to the process of exploration or "free writing" that occurs before one constructs a final draft (Barnes, 1976). In using this terminology from writing to describe classroom discourse more generally, Barnes asserts,

> The difference between exploratory and final draft is essentially a distinction between different ways in which speech can function in the rehearsing of knowledge. In exploratory talk and writing, the learner himself takes responsibility for the adequacy of his thinking; final-draft talk and writing looks toward external criteria and distant unknown audiences. (pp. 113-114, 1976)

It requires courage and modesty to expose one's exploratory thinking to others in the hopes that by engaging in the exchange of ideas in classroom discourse, one might end up with better ideas in the end.

This final segment of the lesson most clearly exhibits what the students had learned about the nature of mathematical knowledge and about their responsibilities as participants in the activity of learning mathematics. When they disagreed with their classmates, the disagreements took the form of presenting evidence that was intended to prove that what had been asserted could not be true. The evidence appropriate to mathematical discourse is an argument that demonstrates that an assertion will lead to a contradiction, given the domain in which the discourse is conducted, or a counterexample. Students were often mustering such evidence in an attempt to defend their own assertions and refute the assertions of others at the same time. Their choice of "refutation" as a form of disagreement with their peers instead of "put downs" or silence-- forms more common to discourse among fifth graders-- is a significant indication of what they believe about how truth is established in mathematics: it is not established by the teacher, or another student, saying that an answer is right or wrong, but by mustering the evidence to support or disprove an assertion, be it one's own, or that of a classmate.

IF THAT IS THE END OF THE STORY, THEN WHAT IS THE BEGINNING? SOME NON-MATHEMATICAL WAYS OF KNOWING MATHEMATICS IN SCHOOL

In order to support the argument that these students had learned something about mathematics, I need to substantiate the claim that they were doing something novel in this discussion, or that they were not doing something that they had been doing before and that students of their age in school do regularly. In this section of the paper I will describe several patterns that characterize students' approaches to learning and knowing mathematics in the classroom before they have learned to participate as active members of a community of discourse about mathematics. Although they are fitting as constructs based on students' experiences with learning mathematics in school, these approaches are in conflict with those that Lakatos, Polya, and others consider to be appropriate to doing mathematics. That conflict gets worked out in the classroom as my students and I negotiate through the year about what our respective roles and responsibilities will be. I take the patterns in student behavior that I will describe here as a starting point for instruction. There are fewer of these behaviors in my classes toward the end of the year than at the beginning.

1. Turning to the teacher or some other reliable authority for ratification

Perhaps the question I am asked most often by observers of my teaching is, "Don't you ever tell them whether their answers are right or wrong?" Students (and many other observers of teaching and learning) expect that this is part of the teacher's role, and they are disconcerted when the teacher does not comply. They ask, "How can I tell whether I am doing well, if you will not tell me whether the answer is right?" or "How do I know if I know my math if you won't check it for me?" Once students figure out that their teacher will not be cajoled into doing the work of establishing the validity of their results, many students look for verification from peers whom they know from previous classes usually get the right answer. Even if they do not ask these students for help directly, they look to them in discussions and feel more confident if their own assertions match those of the students they identify as "smart in math." On occasion, when I have asked a student why he or she revised an assertion at the beginning of the school year, I would be told straightforwardly, "Because that's what Tommy said, and he's usually right."

2. Treating rules, formulas, and facts as if they were arguments

Particularly among students who have been successful at school mathematics because they are good at memorizing and following rules, there is a tendency to use rules as reasons for action, as if they could stand in for explanations. They answer questions about why it seems appropriate to use a particular rule in a given problem with a repetition of the rule or perhaps a reference to the person who taught it to them. These students are generally resistant to and impatient with the representation of numerical calculations in another medium, and they are often unable or disinclined to express the relationship between arithmetic operations and the actions on quantities they are meant to indicate. They are disconcerted when their ability to use a conventional algorithm correctly goes relatively unrewarded by the teacher while students who (from their perspective) have arrived at the "wrong" answer are praised for the questions they have raised or the way they have represented the problem. Once they have arrived at an answer by correctly following the rules, not only do they not consider it their responsibility to listen to or argue with other students' answers, but also, they act in ways that are disruptive to carrying on a discussion.

3. Keeping thinking implicit or private

One common sort of student behavior that does not serve well in the creation of mathematical discourse in the classroom is silence. In particular, silence as a way of expressing disagreement with an assertion is in conflict with the notion that knowing mathematics involves arguing, defending, challenging, and proving one's own ideas and those of others. It is not impossible that these activities could take the form of written rather than verbal communication (as they often do in communication among mathematicians), and that approach would be plausible in a situation where students have the tools, the time, and the skills to express their thinking in writing. But these conditions do not obtain in most elementary and many secondary classrooms.

Often students who have not had much experience with the process of discussing mathematical ideas will answer questions about how they figured something out with phrases like, "I just know" or "I just thought it" or "I don't know how I figured it out." These responses are often delivered in a tone of voice that further suggests, "And besides, it's none of your business or anyone else's how I got my answer." Sometimes these responses are delivered in a tone that indicates that the student does not have the words to tell anyone what mental processes led to a particular conclusion. At other times the student's tone suggests that he or she lacks the courage to expose the thinking behind an asserted answer to teacher and class for comment, or is uncomfortable with having the class pay attention to that thinking. On occasion, this response can be taken as an indication that the student saw the answer on the paper of a classmate at the next table and knows telling that to the teacher that would not be well received, either by the teacher or by the student with the answer.

4. Disagreeing by exerting physical or political power over peers

If students come up with different answers to a problem, it is not unusual for them to try to shout down the opposition, or more indirectly to intimidate someone who disagrees. Students do not come to fifth grade assuming that it is inappropriate to characterize one of their fellow classmates as "dumb" or "stupid" when that person has made what they consider to be an obvious error. Even when they have learned that this sort of talk will not be tolerated in large group discussions, they address their peers that way in small group problem solving activities.

A more civilized variant on this theme and one related to other patterns described above is students' suggesting in a large or small group discussion that we resolve a difference among conjectured answers by voting, without having to listen to why anyone thinks one or another answer is more valid. They argue that voting would resolve the disagreement without exposing the incorrect assumptions or procedures that led to the divergence in the first place. From the perspective of the student who often gets the correct answer, this is sometimes offered as a ploy to push the class to "get on with it" in a way that also gets them some rewards because they can rely on less secure students to vote for their answers.

5. Stubbornness and face-saving behavior

Some students will stick with an assertion even after several arguments have been made against it, saying that the answer is correct because it was obtained by "my way of doing it". They do not distinguish between constructing mathematically legitimate reasons and the act of an individual thinking something up. They act as if they believe that admitting that there is something wrong with their reasoning is an admission that there is something wrong with them. They refuse to expose their assumptions, or they engage in very clever casuistry to explain why their assumptions were valid. They turn the idea that multiple solutions to a problem are possible into the relativistic notion that every solution to a problem should be accepted just because someone came up with it.

These actions on the part of students are only inappropriate if one takes the view that knowing mathematics means engaging in mathematical discourse. Just as we assume that students' constructs of mathematical ideas make sense within the domain of their experience, we need to assume that these social behaviors make sense as an expression of what they think it means to do and to know mathematics in school. If the lesson on exponents had been taught without regard for students learning about the nature of mathematical knowledge, these behaviors would not have been out of place.

CONCLUSION

As I went about trying to teach my students to give up this familiar set behaviors for what Polya calls "the moral qualities of the scientist" I assumed that they would not learn a different way of thinking about what it means to know mathematics simply by being told what to do, anymore than one learns how to dance by being told what to do. I assumed that changing students' ideas about what it means to know and do mathematics was a matter of immersing them in a social situation that worked according to different rules than those that ordinarily pertain in classrooms, and then respectfully challenging their assumptions about what knowing mathematics entails. Like teaching someone to dance, it required some telling, some showing, and some doing it with them, to say nothing of regular rehearsals.

I do not claim that my students' having learned to do mathematics in a way that is congruent with disciplinary discourse is entirely attributable to the social structure I created in my classroom. Students come to any class with varying degrees of expertise and experience in having these kinds of mathematical discussions, some having none and others having a great deal. There are other teachers in the school where I work who also have tried to engage their students in mathematical discourse over the past four years.[14] There is a mathematics coordinator in the district who understands what mathematical discourse is about and believes it to be an appropriate teaching method. My students are taught by several other teachers in different subject matter areas who, in terms of the discourse structures of their disciplines, also expect students to be the authors of ideas. And the students also have parents, of course, at least one of whom is a professional mathematician. But they do act differently toward mathematical knowing at the end of the year than they do at the beginning in my class, and informal questioning suggests that they even carry some expectations about how one does mathematics from my classroom into other settings.

My arguments about what is entailed in teaching students to have a relationship to mathematical knowledge that is more like knowing as it is practiced in the discipline than like conventional school learning draw on work in the history and philosophy of mathematics and from research on teaching and learning in the tradition of social psychology. My organizing ideas have been the "humility and courage" that Lakatos and Polya take to be essential to doing mathematics. I have treated these as social virtues, and I have explored whether and how they can be deliberately taught, nurtured, and acquired in a school mathematics class. The evidence that they can answers my initial research question, but raises deeper questions about what students are actually learning and whether mathematical discourse is what they should be learning in school.

An exploration of these questions would be well served by considering how the activity I have portrayed as "mathematical discourse" looks from other theoretical perspectives, namely: the psychological analysis of individual cognition and the anthropological analysis of the knowledge involved in solving non-school, non-disciplinary problems. From these perspectives, we might ask: What is to be gained from students learning to do mathematics with humility and courage, from their learning to generate knowledge in argument with one another in a manner similar to the way it is generated in the discipline? Are students' individual conceptual structures are enhanced by this sort of activity? Is the mathematical knowledge acquired in the context of mathematical discourse more usable for the solution of the sort of unspecialized and messy practical problems that students will face once they leave school? How is learning about mathematics as a social construction in a community of discourse different from learning to do mathematical reasoning by interacting with a computer? or a tutor? How is learning mathematics in the elementary school classroom distinguished from learning it from an apprenticeship in a community of working mathematicians?

Further questions follow if we ask whether mathematical discourse should occur in school from various practical perspectives: What do you need to know to teach this way? How is that knowledge acquired? How much more time and energy does this kind of teaching demand of teachers? What other resources besides skill and knowledge might teachers need to do this kind of teaching? How can the results of such teaching be portrayed to the public in such a way as to communicate that teachers are teaching and students are learning what they need to learn? Knowing what mathematical discourse might look like in the fifth grade, and establishing that it is possible for it to happen there, is only the beginning of what looks to be an interesting but difficult exploration.

NOTES

[1] The author would like to acknowledge the contributions of Susan Florio-Ruane and the members of The Teachers' Study Group on Mathematics in East Lansing, Michigan to the development of the ideas in this paper, and the comments of David K. Cohen and Deborah Ball on earlier drafts.

[2] The common definition of axiom is "self-evident truth" and for a long time, axioms were regarded as such in mathematics. It was not until logicians found that the same set of self evident truths could lead to contradictory conclusions, rather than certainty, that this view came into question in the discipline. See Kline, 1980; Lakatos, 1976, p. 56.

[3] They are also different from the classical logicist or Platonic view of mathematics as the discovery of ultimate truth that can not be refuted because of its logical foundations. See Hadamard, 1947: Hoffman, 1987; Hardy, 1940.

[4] The study reported here is part of a larger study, "Teaching Mathematics for Understanding and Understanding Mathematics Teaching," funded by the Spencer Foundation through the National Academy of Education. The data are drawn from my own classroom teaching. I have taught fourth and fifth grade mathematics during the past six years, collecting data on both teaching and learning during three of those years. These data included audio tapes of lessons for six months, video tapes of two units, records of speech and visual communication kept by an observer at least three times a week over three years, notebooks in which students did their daily work including writing and drawing they do to represent their thinking, and students' homework papers. I have also kept my own detailed field notes on lessons, including descriptions of how lessons and units were planned and implemented and initial analyses of the planning process itself, the lessons as they were taught, and students' work. I have further analyzed a portion of these data using triangulation among different data sources and constant comparison within and between lessons.

[5] The application of such ethnomethodological and sociolinguistic frameworks to the study of mathematics teaching and learning has also been advocated by Bauersfeld (1979) and Steiner (1987). Both argue that mathematics learning is a process of social as well as individual construction, and that patterns of interaction are powerful in shaping both students' beliefs about what doing mathematics means and the sorts of activity they are inclined to engage in during a mathematics lesson.

[6] Being both the analyzer of the teacher's role and the incumbent, my use of first and third person in this paper is admittedly schizophrenic.

[7] An example of how this hypothesis testing works in relation to a more conventional topic in the upper elementary curriculum-- comparing the relative magnitude of two decimal numbers-- is described in Lampert, in press a.

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8 I also initiated them into the use of conventional mathematical tools and language. For a more detailed explication of this aspect of the teachers role, see Lampert, in press a and Lampert, 1986a, 1986b.

9 These roles are similar in many ways to those described by Palincsar and Brown (1985) in their presentation of "reciprocal teaching". Following the Vygotskian notion of "scaffolding" the teacher's participation fades somewhat as students become more expert in the pattern of discourse that is being taught. A difference between teaching mathematics and teaching reading comprehension that seems worth exploring has to do with the interaction between learning content and learning discourse structures. At issue is the question of how much and what sort of mathematics a teacher would need to know to teach content using this method (Ball, 1988). Work currently in progress on the explicit teaching of strategies to enable students to become more independent learners of mathematics has focused on students learning problem solving strategies rather than on teaching and learning what is conventionally thought of as mathematical "content" (Brown, 1988; Resnick, in press). Some studies of group problem solving have addressed learning in school mathematics as a function of dialogue in small group or paired interactions (Noddings, 1985; Schoenfeld, 1985), and Schoenfeld has explored the teacher's role as a more expert problem solver who models "ways of knowing" in college mathematics classes (Schoenfeld, in press).

10 The use of orders of magnitude for thinking about relationships among numbers is attributed to Archimedes; his notions about comparing magnitudes by comparing orders of magnitude, or "powers" later evolved into the scientific notation and logarithms that are indispensable to applications of mathematics across the spectrum of scale, from microbiology to astronomy (Jacobs, 1982). Before Archimedes, the idea of actually counting something like the number of grains of sand in the universe (which is what Archimedes set out to figure out how to do) was inconceivable; Archimedes' tool made it seem doable. Several centuries later, another intellectual tool was invented which further enabled the use of orders of magnitude in mathematical problem solving. Descartes is attributed with the first use of the exponential form "x^2" to symbolize "x multiplied by x". This was more than a mere shorthand, for it provided a basis on which new mathematical relationships could be conceived and represented and then operated upon with much simplified procedures. The concurrent development of logarithms by Napier led to entire new branches of mathematical theory and powerful scientific applications of quantitative tools (Kramer, 1970). See Cassirer (1957) for an explication of how these symbolic tools function in the development of human knowledge.

11 After each part of the lesson is described, I will attempt to explain some aspects of my teaching, but these *post hoc* representations of my thinking are not meant to suggest what I may have been thinking at the time. Reflective justification is not the same thing as making reasoned choices about what to do within the act of teaching lessons. Actions may be warranted after the fact by arguments from mathematics, psychology, or pedagogy, but these warrants are not the same as reasons for doing one thing or another. In practice, teachers often are choosing between two or

more courses of action, each of which is sufficiently warranted, but any one of which might be in logical conflict with another. See Lampert, 1985; Buchmann, 1988; Ball, 1988.

[12] The assumed domain for exponents is whole positive numbers at this point in the discourse.

[13] A recent paper by Cobb, Yackel, and Wood (in press) examines the teacher's role in shaping the way students interpret their feelings about participating in mathematical discourse in the classroom, with a focus on the students' developing capacity to distance themselves from their own activity to attempt to understand alternative points of view and resolve conflicts between incompatible solutions.

[14] I have taught other teachers to use the strategies I have described in this paper, and they are using them at different grade levels throughout elementary and middle school (Lampert, in press b). Studies of their teaching practice are underway (e.g. Parker and Weinberg, 1988).

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The Situated Activities of Learning and Knowing Mathematics

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As a context for the research that I will describe, I want to focus attention on a view of the goals of mathematics education. This view emphasizes two aspects of mathematical knowing, its active character and its occurrence in situations.

If we view mathematical knowing as an activity, we focus on what we want students to be able to do as a result of their learning. We hope that students will be able to reason effectively about mathematical concepts and principles, and will be able to reason with mathematical concepts and principles about the events and systems that they encounter in their lives. On this view, learning mathematics involves acquiring aspects of an intellectual practice, rather than just acquiring some information and skills. Some aspects of this view are highlighted if we think of mathematical learning as a kind of cognitive apprenticeship, where students are learning to engage in a kind of activity that is modeled and exemplified in the classroom setting (Brown, Collins & Newman, in press; Lave, Smith, & Butler, in press). The view is also informed by considering mathematical knowledge as it is embodied in the practice of the mathematical community (e.g., Kitcher, 1984; Toulmin, 1972), rather than as it is encoded in mathematical writings.

The second aspect that is emphasized is the situated character of mathematical knowing. Like all cognition, mathematical reasoning and understanding occurs in situations. In a view that is emerging from studies of expertise (e.g., Dreyfus & Dreyfus, 1986), analyses of technology in social

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contexts (e.g., Suchman, 1987; Winograd & Flores, 1986), and ethnographic investigations of cognition in work (e.g., Carraher, Carraher & Schliemann, 1985; Lave, 1988; Zuboff, 1988), some of us are becoming convinced that we need to treat the concept of knowing relativistically, in something like the way that motion is relativistic in physics. That is, the properties of someone's knowing cannot be specified independently of the situation that provides a frame of reference for the person's cognitive acitivity, such as a physicist cannot specify properties of the motion of an object, independently of a frame of reference. The knowing of mathematics is situated in social and intellectual communities of practice, and for their mathematical knowing to be active and useful, individuals either must learn to act and reason mathematically in the settings of their practice, or they must acquire capabilities to generate mathematical meaning and solutions of problems in situations that they encounter.

If we view mathematical cognition in this way, we are led to a pessimistic assessment of most current educational practice. A mathematics classroom is itself a determining context for the activities of students. The activities that dominate the practice of learning by students in classrooms are listening to what an authoritative teacher tells them, watching the teacher perform procedures, and learning to mimic the procedures that the teacher and textbook present. Successful learning includes adaptation to the requirements of the learning environment, and often does not go beyond such adaptation. Students who adapt successfully to the requirements of mathematics classrooms learn to listen, watch, and mimic effectively. If we believe that these receptive forms of activity are insufficient as outcomes of mathematics education, we may be forced to consider fundamental changes in the kinds of activity that students engage in as they learn mathematics.

Symbols and Meaning in Mathematics

One problematic aspect of learning mathematics results from the strong role that symbols play in mathematical reasoning. Activities in mathematics classrooms are dominated by manipulation of symbols. These symbols are intended to have meaning, of course, but they refer to abstract entities that are absent from the cognitive worlds of students.

To think about these issues, it is helpful to frame them in terms of some general concepts in the theory of symbols and representation. Figure 1 presents some of the relevant distinctions, in a form first presented by Smith (1983). The two rectangles on the left are the domain of symbolic expressions. The lower rectangle, labeled "symbolic notations," has sequences of characters that are well-formed according to the syntax of mathematical expressions. The upper rectangle, called "symbolic structures," has the expressions as they are organized by someone who understands them, including their constituent structure. <theta/s> is a function that maps sequences of characters to structured expressions, and is approximately a parsing function.

The rounded rectangles on the right are the domain of things that the expressions are about. The lower rounded rectangle has objects and events -- the things that exist and the things that happen. The upper rounded rectangle has organized versions of those objects and events that are labeled "structured objects and events." The structures of objects and events include properties such as what things are parts of what other things, which objects are members of sets along with which other objects, which actions and events are parts of other actions and events, and so on.

<psi> and <mu> are functions that take structures to other structures. <psi> is a function that transforms symbolic expressions into other

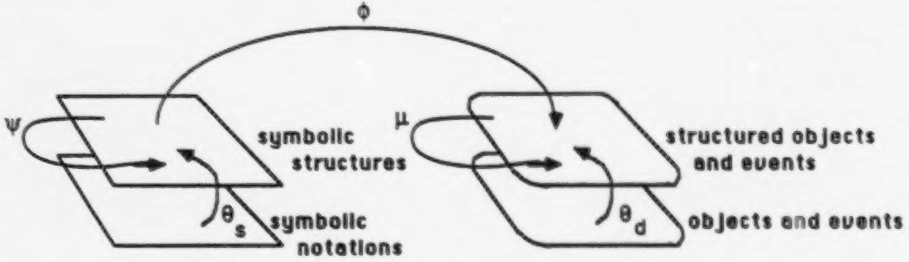
ϕ
ψ
μ
symbolic structures
θ_s
symbolic notations
structured objects and events
θ_d
objects and events

Figure 1. A view of semantics (after Smith, 1983).

expressions. Operators in <psi> include rules of transformation between equivalent expressions or equations and rules of inference that justify writing new statements in a proof. <mu> is a function that transforms situations with objects and events into other situations. Its operators are the actions and processes that can occur in the world.

<phi> is the most important entity in the diagram. It is the mapping from the symbolic expressions to the objects and events that the expressions are about. For example, if there is an equation on the left side that is supposed to be about a relation between quantities in some process in the world, then the correspondence between the terms of the equation and the quantities involved in the process should be included in <phi>. A requirement on the construction of <psi>, with an intended interpretation <phi>, is that when we perform operations on symbols that correspond to things that happen in the world, the symbolic expressions that result should describe the states that result from the real events.

Figure 2 shows a more elaborate version of the scheme, needed for thinking about mathematical systems. Sometimes we use mathematical expressions to describe processes and events in the world, as in the equations we use for word problems. Other times we use mathematical expressions to describe abstract entities and processes, such as sets, numbers, variables, and functions. Relations between concrete and abstract entities are much too complex to be captured by a pair of domains connected by an arrow, but this diagram at least recognizes that some of the uses of mathematical expressions involve denotations that are not directly in the physical world.

The goals of mathematics education are something like Figure 2, in which students know what the symbols can refer to in situations in the world, as well as in the domain of mathematical concepts and principles. In fact,

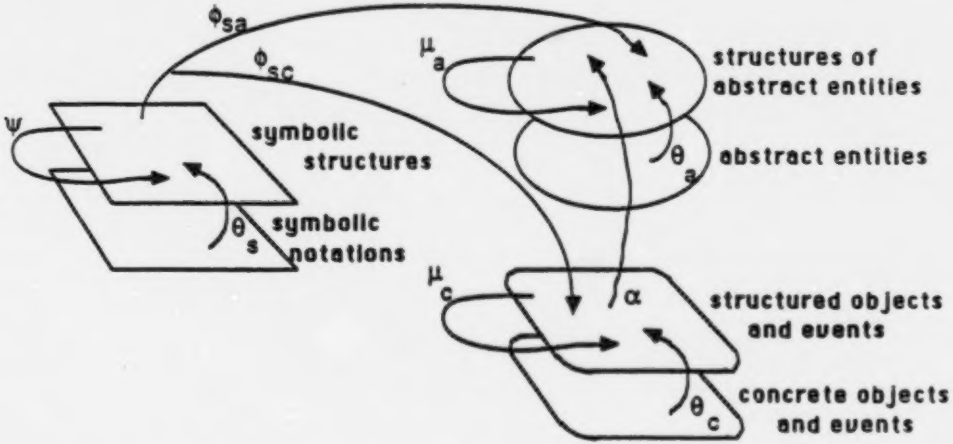
ϕ_{sa}
ϕ_{sc}
μ_a
structures of abstract entities
ψ
symbolic structures
abstract entities
θ_a
θ_s
symbolic notations
μ_c
α
structured objects and events
θ_c
concrete objects and events

Figure 2. A view of semantics with abstract entities

getting something like Figure 2 into students' minds is not enough. We want them to be able to construct the <phi> relations for situations and concepts that they encounter. Unfortunately, all of this is probably beyond what most students acquire in their mathematics instruction. The outcome of school mathematics learning for most students is more like Figure 3, where the only meaning that symbolic structures have is in the symbolic notations themselves. The symbols are supposed to denote physical quantities or conceptual entities, but their referents are just notations -- marks on paper or a chalk board. This is a perverse kind of knowledge, and a gross distortion of the epistemology of mathematics, but there is considerable evidence that many students come away from their classroom experience in mathematics with little more than a set of rules for manipulating symbols, which they can verify only by comparing the answers they get to symbolic problems with answers that are prescribed by teachers and textbooks.

Beliefs and understandings about mathematical knowing.

As bad as these misconceptions about the meaning of mathematics are, they may not be the most serious consequences of our present mode of instruction. There are broader issues of mathematical epistemology for which current instructional practice has equally negative effects that are even more worrisome than those of mistakenly believing that mathematics is about its symbols.

The issues have been studied by Belenky, Clinchy, Goldberger, and Tarule (1986) and by Dweck (1983; Dweck & Leggett, 1988). They involve beliefs and understandings that individuals have and come to have about themselves as knowing agents and about their capabilities for learning and knowing.

Belenky et al.'s study focused on the epistemological positions of

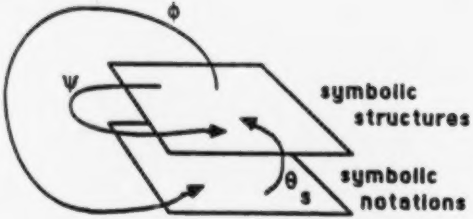
ϕ
ψ
symbolic structures
symbolic notations
θ_s

Figure 3. An unintended outcome of "good" instruction.

individuals engaged in different kinds of learning institutions, including colleges and human service agencies that provide assistance and information for parents. They distinguished several epistemological positions, involving quite different orientations toward learning and knowing. In one quite primitive orientation, called received knowing, one only obtains knowledge from authoritative sources. Learning is passive and assimilative, and the warrants for knowledge are the alleged authority of sources. Belenky et al. contrasted received knowing with more active epistemological positions, including subjective knowing where the individual passes information and opinions through a personal filter, procedural knowing where the individual understands knowledge to be the product of cognitive or social processes, and constructive knowing, which results from a flexible combination of cognitive and social processes.

Dweck's research (e.g., Dweck, 1983; Dweck & Leggett, 1988) has also been concerned with individuals' beliefs and attitudes about knowing and learning. She has studied children's beliefs about what is involved in being "smart," and has related these beliefs to the children's causal attributions regarding their successes and failures, and to their orientations and tendencies to engage in intellectual tasks. Dweck's research shows that some children understand intelligence as a fixed entity, while other children understand intelligence as a set of capabilities that they strengthen by engaging in challenging intellectual tasks. The entity theorists view intellectual tasks as occasions in which their (fixed) amounts of intelligence will be evident to other persons, and are attracted to easy tasks in which they can display a high level of performance. When they fail, they tend to blame their low level of ability, and they tend to explain their success on easy tests or luck. The malleable-trait theorists view intellectual tasks as opportunities for cognitive growth, and are attracted to challenging tasks in which they can improve their skills. When they fail, they tend to blame the tests or other situational factors, and they explain their success by their high ability and outstanding effort. These patterns are summarized as helplessness and mastery, respectively, by the children whose beliefs involve fixed entities and malleable traits, respectively.

There has been very little research addressed to the question of how individuals develop the epistemological beliefs and understandings that they have. Even so, it may be reasonable to speculate on the effects that different classroom practices may have on the epistemologies that children develop, both at a general level and with respect to specific domains of knowledge, such as mathematics.

In most school instruction, what students do is mostly to listen, watch, and mimic things that the teacher and textbook tell them and show them. If students' epistemologies are influenced at all by the experiences they have, then most students probably learn that mathematical knowledge is a form of received knowledge, not something that is constructed either personally or socially. Another probable outcome for many students is a belief that they were endowed with a low level of mathematical ability, and that there is little or nothing they can do to become mathematically able. The links between classroom experiences and students' personal epistemologies undoubtely are subtle and complex, and classroom experience undoubtedly is just one factor in determining what students believe about mathematical knowledge and about themselves as mathematical knowers and learners. Even so, it seems likely that the influence of current classroom practices on the epistemological beliefs and understandings of most students is largely negative.

The picture that I have drawn of ordinary classroom instruction is rather bleak. If things are this bad, how is it that any students learn and understand mathematics successfully? One possibility is that the teachers and students who achieve significant learning do so in spite of the situation that they are in. There is a natural human tendency to make things meaningful if it is at all possible to do that. Some teachers and students come to the classroom with strong beliefs and desires that mathematical symbols must mean something, and they manage to find ways to construct meaningful knowing of mathematics, even with the weak support of meaning that is provided in the curriculum and the learning situation. According to their personal epistemologies, mathematical symbols are meaningful, and they act according to these beliefs, even if they do not have explicit conceptual means for expressing the beliefs. Those few students who succeed in constructing meaning in spite of the routine character of mathematical knowledge in the classroom probably see themselves, correctly, as exceptionally talented, and see mathematical knowledge as a product of lonely and idiosyncratic creativity.

A Situation for Knowing and Learning about Functions

Proposals for sweeping reforms of mathematics education have been made before, and will have to be made many times in the future. The task of changing anything as complicated as mathematics education requires efforts of many kinds. One kind of contribution is in research that clarifies some of the capabilities that students need if they are to engage in meaningful mathematical activities in their educational practice. I will describe some preliminary results of a research project that I hope will provide such a contribution. Individuals who are collaborating in this research include

Joyce Moore, Meg Korpi, Jean-Luc Gurtner, Greg Pribyl, and Jon Simon.

The mathematical topic of the research is the concept of function. Several researchers have shown that many students do not achieve good understanding of this concept when they study algebra. A few of the relevant studies have been done by Dreyfus (1983), by Greeno, Magone, Rabinowitz, Ranney, Strauch, and Vitolo (1985), and by Janvier (1978); and Schoenfeld (see Schoenfeld, Arcavi, & Smith, forthcoming) is conducting studies with which the research that I will describe is coordinated.

The research is focused on three general questions, each of which has a theoretical side and a side related to educational practice.

The first question is the nature of students' implicit understanding or conceptual competence regarding variables and functions. The theoretical side of this question in cognitive science relates to other topics in which children have been shown to understand concepts and principles implicitly for which they have significant understanding but are not able to describe or explain the concepts and principles in language. For example, Gelman and her associates (Gelman & Gallistel, 1978; Gelman & Meck, 1986) have shown that preschool children have significant competence regarding principles of number and counting, contrary to impressions that were based on findings by Piaget and others (e.g., Piaget, 1941/1965). One goal of our research is to characterize implicit understanding that students have regarding functions and variables, which would show the other side of the coin from negative findings that I have mentioned. In fact, a positive expectation for students' intuitive understanding of functions is supported by work on functions by Piaget, Grize, Szeminska, and Bang (1968/1977).

The practical side of this question involves the feasibility of learning in which students discuss and refine mathematical concepts and principles that are important for understanding algebra, rather than receiving them. If students have a reasonably rich set of intuitions about functions that they can show if appropriate situations are provided, it makes sense to think about ways to develop and refine their understanding through their collaborative discussions. If their intuitions about functions are very sparse, it might not be worth much effort in trying to develop such instruction.

The second question involves characteristics of situations in which implicit understanding of functions is effective. Earlier, I discussed a view that cognition is relativistically dependent on situations. On this view, the analysis of situations that enable effective reasoning is a fundamental question for cognitive science. In the research that I discuss here, the question involves identifying characteristics of a situation in which students can reason effectively about functional relations among quantities.

The practical side of the question about situations involves possible designs for materials and activities that could be used in instruction in algebra. In current instruction, algebraic ideas are taught almost entirely using formulas and operations on them. Graphs and word problems could provide referential meaning for expressions and equations, but for most students, graphs and word problems are difficult to relate to expressions and equations, and they produce more difficulties of understanding, rather than cognitive support. If situations can be found in which students reason easily and effectively about functions and variables, these situations could be used to anchor students' understanding of algebraic notation. We might say, then, that this research could result in identifying materials for algebra that play the role that place-value blocks, cuisinaire rods, and fraction pie diagrams play for more elementary mathematics.

The third question that is addressed in this research is about language.

Students' reasoning is a social process, and both depends on and is expressed by language. The issues about language in this research are not issues about students' understanding of formal mathematics. Rather, they involve uses of terms in ordinary language that refer to quantities that are properties or relations in the situation that we use to exemplify functions.

In the usual view of language, words are considered to have meanings, and communication depends on shared knowledge of the meanings of the words. In an alternative view, which is a leading part of the developing relativistic view of cognition, important aspects of meaning depend on the situation, and are determined in a process of cognitive negotiation between the individuals who are communicating. There is a cliche in linguistics: "Words don't mean, people do," which captures the gist of this idea. This relativistic view of word meanings is not the solipsistic view of Humpty Dumpty, who said, "I can make a word mean anything I want to." For communication to work, the meaning that is arrived at in a group of people has to be agreed to by the members of the group. It also doesn't imply that people shouldn't work to make their language clearer and more precise by setting up definitions of terms that they then apply when they communicate, at least in certain settings. But it does say that even when terms have been defined explicitly, there will be important aspects of meaning that are determined collaboratively, by a kind of negotiation, in contexts of communication.

In our research study, students were asked to talk about a machine that was novel to them, and there were many occasions in which meanings of terms had to be negotiated. These were particularly interesting when they dealt with quantitative properties of the situation, because these often reflected concepts that the students understood implicitly that they had to work to express in language. The collaborative effort to communicate about the events in the situation contributed to the students' understanding by providing a set of words that represented their ideas more explicitly and clearly.

The practical side of this issue relates to the role of language in learning. In teaching mathematics, we often begin by giving students definitions of terms, and then we expect them to use the terms correctly because "they should know what the words mean." This is probably a wrong-headed way to think about language in learning. Formal definitions have an important role in mathematics, of course, but they are not the main ways in which terms acquire meaning for communication. Instead, we need to create situations for students to communicate about with each other and with their teachers. The process of communication is an important vehicle for developing more articulate forms of understanding, and may be the main cognitive resource for developing general forms of knowing.

Method

In our research, we ask students to reason about the operation of a simple machine, sketched in Figure 4. The machine is a board, about a yard long, with two grooves that run lengthwise. Beside each groove is a ruler, marked in inches. Each groove holds a small metal block with a pointer on its top surface. A string is tied to each block. At the end of each groove there is an axle attached to a wheel with a handle so the axle can be turned. Metal sleeves can be placed on the two axles, and the sleeves differ in circumference. The two axles can be linked together so they turn simultaneously, or they can be separated so they turn independently.

The device is an embodiment of linear functions. Think of the formula $y = mx + b$. y is where the block is after you turn a handle some number of times. b is where the block was when you started counting the turns. m is

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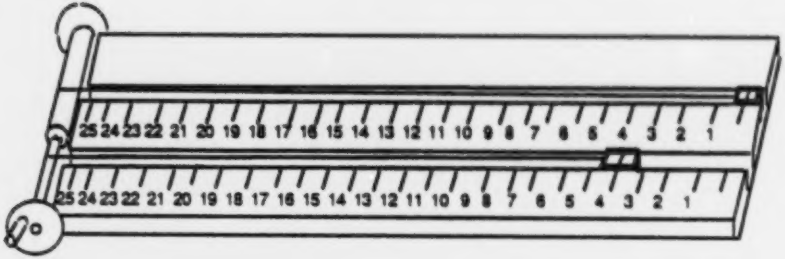
25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Figure 4.

the circumference of the spool you used. And x is the number of times you turned the handle. As we have used the machine, the functions all have positive slopes of 2, 3, 4, or 6, with values of y between zero and 30 or so.

In terms of Figure 2, we designed the machine as a concrete physical system, with the main physical operation of pulling blocks along the tracks by turning the handle or handles. We hoped that functional relations between the quantities in the situation (e.g., the relation between number of turns and distance moved) would be obvious to students, and would correspond to some abstract entities that they would reason about. In our first study, we made no effort to include the standard symbolic expressions of algebra. Some students made use of mathematical symbols, usually in the form of tables. Mainly, though, the symbolic expressions involved in our situation are in ordinary English.

In a preliminary study, we interviewed students in pairs, either two girls or two boys. In most cases, students who were interviewed together were friends. The interviews were about 40 minutes long. First, the students became familiar with the machine; they used different spools and starting points for a single track, and they talked about what happened. In the second part of the interview, situations were set up by the interviewer with blocks started at specified places in the two tracks, spools of specified sizes, and the axles specified to be linked or independent. For each situation, students were asked three open-ended questions: (1) What will happen when you turn the handle (or handles)? (2) What can you say about how far apart the blocks will be? (3) What can you say about where the blocks will be when you stop turning?

In the third part of the interview, situations were described and the students were asked how to make them happen. First, students were asked how to make one of the blocks get to 20 before the other block. Next, they were asked how to make the two blocks get to 24 at the same time. In each case, they were asked to give three solutions, with constraints added to prevent the same combination of equal or unequal variables.

In the fourth section of the interview, the students were asked a series of questions requiring inferences based on the linear functions in some situations. For one question, the red block had a 3-spool and started at 10, the blue block had a 4-spool and started at 5, and the axles were linked. Students were asked how many turns it would take for the blue block to catch up with the red block. For another series of questions, the red block had a 3-spool, the blue block had a 6-spool, the axles were linked, and the blocks started at zero. Students were asked whether the red block would ever get ahead, how far ahead the blue block would be after four turns, whether the blue block would ever be ahead by six inches, whether the number of the blue block's position (called "the number the block is at") would ever be twice the number of the red block's position, or three times the number of the red block's position. A similar series of questions was asked for a situation with the same spool sizes, but with the red block started at nine and the blue block at zero.

Finally, students were asked a question designed to elicit discussion in general terms: "Imagine that blue is going to start behind red by some amount, but I won't tell you the amount, and blue has a bigger spool than red, but I won't tell you the spool sizes. The axles have to be linked. What can you say about when blue will catch up with red?"

We had interviews with ten pairs of students. Four pairs were seventh-grade students who had not studied algebra at all. Three pairs were ninth-grade students at the end of their first course in algebra. And three pairs were eleventh-grade students at the end of their second course in algebra.

Some preliminary findings

I will briefly summarize the performance of the ten pairs of students on one of the sets of questions in the interview to provide a sense of the overall level of performance and variability in our data. These questions were in the fourth section of the interviews, and asked quite specifically for inferences about values of variables. The setup for these questions had a 3-spool for the red block and a 6-spool for the blue block, and both blocks were started at point 0, the far ends of the tracks. Mathematically, the situation can be described as two functions of a single variable: $f1(x) = 3x$ (the red block) and $f2(x) = 6x$ (the blue block). Our interview protocol called for the interviewer to go through the following questions, to the extent that time permitted.

a) When you turn the handle, will the red block ever get ahead? If students say "yes," ask when that will be. If students say "no" ask why not.

b) How far ahead will the blue block be after four turns?

c) Will the blue block ever be ahead by six?

d) Think about the numbers that the two blocks are at as they go along. Will the number that blue is at ever be two times the number that red is at? If students say "yes," ask when that will be. If students say "no, ask why not.

e) Will the number that blue is at ever be three times the number that red is at? If students say "yes," ask when that will be. If students say "no," ask why not.

These questions were asked during the fourth section of the interview, and students had been working with other questions (most of them more openended than these) for about 30 minutes.

A correct answer to question (a) recognizes that $f2(x)$ is greater than $f1(x)$ for all $x > 0$. All the student pairs who were asked gave correct answers (the question was omitted for one ninth-grade and one 11th-grade pair). A mathematical explanation would be that $6x > 3x$ (for $x > 0$) because the values of linear functions with zero intercept are proportional to their slopes, and the slope of $6x$ is greater than the slope of $3x$. In the machine, the sizes of the spools corresponded to the slopes of the functions. The explanations that students gave identified the difference between the spools as the reasons for the blue block being ahead of the red block consistently. One of the 11th-graders said, "Because right off from the start, after the first turn this one (touch the 6-spool) has a three-inch lead, so it (the red block) can never get ahead. Every turn it'll gain three inches." One of the seventh-graders said, "Because this (the blue-block side) is bigger and it's a bigger sort of rotation; it will be longer; it will wind up more string than this one (the red-block side). In fact this one will just go more ahead."

Question (b) could be answered by evaluating both functions $f2(x)$ and $f1(x)$ with the value $x = 4$ and comparing their values. In finding values of $f1(x)$ and $f2(x)$ they could both be evaluated directly, or one could be evaluated and the other found by using the function $f2(x) = 2f1(x)$ or $f1(x) = (1/2)f2(x)$. Another possibility is to consider the function $f2(x) - f1(x) = 3x$ and evaluate that function at $x = 4$. Of eight pairs of students who were asked question (b) (it was omitted for two 11th-grade pairs) four pairs of students gave the correct answer initially, and three more pairs gave a correct characterization of the inference when they had observed the event occurring in the situation. Three of the four seventh-grade pairs and the 11th-grade pair answered "12 inches," and then gave appropriate explanations. In three of these cases a student inferred that one block would go either to

12 or 24, then inferred the other block's position as double or half of the first block, and inferred the difference between the positions. One of the seventh-grade pairs only said "three, six, nine, twelve," which might correspond to inferring the position of the red block, or it might correspond to inferring the difference between the blocks which starts at zero and increases by three on each turn. In one of these cases, student ST2 gave the correct answer quickly, saying "Six times four is 24 and half of 24 is 12." Then the following exchange ensued:

> ST1: I have something else: One turn is three inches, and three times four ---
> ST2: That was on a different thing, though.
> ST1: Let's see -- (to INT) what was it we were trying to get?
> INT: The question was, how far ahead will the blue be after you make four turns. And they're linked.
> ST1: Ah, I think 11. It's so close -- I don't want to (motion to ST2).

[We think that ST1 thought it would be better not to give the same answer as ST2 had already given.]

> INT: Oh, you can copy him; I mean, I want you guys to work together.
> ST1: Ok, I say 12 too, then.
> INT: (to ST1): How did you figure, did you figure out 12, you think, a different way than he did?
> ST1: Yeah, well, I figured out three, ok, one turn is three, and then --
> INT: Three for what?
> ST1: One turn it'll go three inches.
> ST2: <ST1's name>, that was for a different thing. You don't know what it is for a six.
> INT: Well, the six is six inches, for a turn.
> ST1: Oh, it is? Oh yeah, huh. So six times -- (laughs) -- uh, 12.
> ST2: I figured out six inches -- six divided by three is two, um, and then 12 times three, I mean 12 times two is 24 -- I don't know how I did it (shrugs).
> INT: Ok.
> ST2: Let's try it. (turns handle)
> INT: Are you counting turns?
> ST2: Oops. (ST2,ST1, and INT reset the machine, run the trial again.)
> ST1: I think, in four turns? 24. I think 24.
> INT: 24 apart?
> ST1: [something inaudible]
> ST2: Ah, now we changed it.
> ST1: 24.
> ST2: Ok, 24? (Count, pointing to positions from blue's position at 24 back toward red's position at 12) one, two, three, four, five, six, seven, eight --

ST1: Oh, like that!
ST2: nine, ten, 11, 12, 13, about -- 12, maybe.
ST1: Oh, I thought 24, about how far it would go.
INT: Ok, so you were --
ST1: I didn't know it was 12 in between. I thought it was -- who would get to 12 first, in four turns. Oh, no!

One seventh-grade pair began by guessing that the blue block would be ahead by "um, maybe, four or five inches," or "maybe around three," but when the interviewer showed them a single turn, one of the students said, "That's three, and that's about six, and that's, that's about three inches ahead, so it was twice as fast as the other one, so when it got to nine, when this one (the red block) got to nine, this one (the red block) would be at 18." When the interviewer asked, "So if I made four turns, how far ahead would this one be?" the student said the blocks would be at 12 and 24, and when the interviewer asked, "So how far is this one ahead of that one?" the student said, "It's twice as fast, it'll be twice as much."

The answer to question (c) could be obtained in symbolic algebra by solving the equation f2(x) - f1(x) = 6 or 6x - 3x = 6. It also could be obtained by considering that the difference f2(x) - f1(x) is 3x; hence, solving 3x = 6.

Nine of the student pairs were asked this question, and seven of the nine gave correct answers. One of the unsuccessful pairs was in seventh grade; the other was in ninth grade. In two of the successful cases (one in 7th grade and one in 9th grade) the answer was apparently found by determining where the blocks would be after each of the first two turns, and noticing that the difference after the second turn was six. One student in an 11th-grade pair apparently recognized that the two-to-one ratio would produce a difference of six after two turns. In one of the seventh-grade pairs the one student apparently used the two-to-one ratio, and the other apparently inferred the positions of the blocks after each of the first two turns. One ninth-grade student and one 11th-grade student apparently used the fact that the difference between blocks started at zero and increased by three each turn; therefore, the difference after two turns would be six.

One of the ninth-grade pairs recognized the two-to-one ratio, and that the difference on each turn was three, but denied that there would be a difference of six. A student said, "It wouldn't because if it moves it three it would always be ahead by half; because if it moves three and six then there's a differences of three." One possibility is that the student understood "be ahead by six" to refer to the difference in distance on a single turn, rather than a difference in position after some number of turns. The interviewer asked, "Is there any number of turns you can make that will ever make the blue block ahead by six?" but the student indicated not. In the unsuccessful seventh-grade pair, the student who was asked initially by the interviewer indicated that the question was hard to understand, and the interviewer guided the student to the answer; there were some indications of understanding along the way, but they were not entirely convincing.

A correct answer to question (d) could be obtained by evaluating the two functions f2(x) and f1(x) at any point. The answer could also be based on recognizing that f2(x) = 2f1(x) is true for all x. Question (e) calls for recognition that the ratio of f2(x) tof1(x) yl is always two-to-one; therefore, it would never be three-to-one.

All of the student pairs who were asked question (d) answered it correctly, and all the pairs except one showed that they understood the proportionality and universality of the relation. (At least one of questions (d) or (e) was asked of all the pairs except two of the 11th-graders.)

As an example, one of the seventh-grade pairs answered the question, "Will the number that the blue one is at ever be two times the number that the red one is at?" by saying "Always." Then one of the students said, "When they're linked, and they're both starting at the same place, always." To the question, "Will the blue one ever be at a number that's three times as big as the read one?" ST2 said, "No." ST1 said, "If you had different spool sizes -- -" and ST2 said "The way the situation is right now, it'll always be two times." Finally, ST1 said, "If they're not joined, then you might be able to ..., or if you changed the spools."

Answering question (e), one of the ninth-grade students said, "No; because the blue spool is twice as large as the red spool, so it can't move any faster than twice as far."

A relatively weak answer to (d) and (e) was given by a pair of ninth-graders. These students were were unsuccessful on both questions (b) and (c) in this set. They gave the following interchange:

INT: "Will the number the blue is at ever be twice as big as the number the red is at?"
ST2: "Yeah"
INT: "I heard a yes."
ST1: "With the two and the four."
INT: "How about with the three and the six?"
ST1: "I don't know."
ST2: "Yes, because this is twice as big as this."
INT: "So where will it be twice, the blue be at twice as big a number? Like in how many turns?"
ST2: shrugs shoulders
INT: "Not sure?"
ST2: shakes head no
INT: "Ok, let's try this one. Will the blue ever be at a number which is three times as big as the number the red is at? Can you think out loud?"
ST1: "I don't know how to figure it out, so -- what did you ask, three times?"
INT: "I was wondering if the blue, if the number the blue is at will ever <never?> be three times as big as the number the red is at?
ST1: "Well, no."
INT: "It won't be? How come?"
ST1: "Because it's always half. If you move the six and then you move the three, and then you move it again, then that's 12 and the next one's nine, so there is always a three-inch difference."
INT: "There's always a three-inches difference between?
ST2: "Yeah -- no, it's a half. I'm not sure."

Discussion (with some additional examples)

Implicit understanding of functions. The first question that the research was designed to address is whether students would show significant understanding of functional relations among quantities in the situation, thereby supporting the inference that they have implicit understanding of the concept of function. Most of the questions in our interviews dealt with that question in a relatively complicated way, involving comparison of two functions rather than inferences based on just one. This was because in pilot data, most of the information needed to answer questions about individual functions seemed obvious to the students, and they were much more interested when we began asking about pairs of functions. We also reasoned that if students showed understanding of relations between pairs of functions, we could infer that they were capable of understanding individual functions.

To illustrate students' capabilities of reasoning about a single function, I present part of a protocol from a pilot interview, where a pair of eighth-grade students was asked a sequence of questions about a single function. There was a 6-inch spool on one of the axles. The interviewer was suggesting that one person can give another person different verbal instructions:

INT: You could tell him to turn it until it got to --
ST1: the end
INT: the end
ST2: 30 inches or something
INT: or you could tell him to go until it got to 30
ST1: Are you going to try it? Let's see if this really works.
ST2: Are you going to turn it 5 times?

ST2's quesyion indicates a clear sense that the number of turns and the final position are functionally dependent; in fact, ST2 implicitly solved the equation "$6x = 30$," without being asked to.

After some intervening conversation, the interviewer asked

INT: ...could you make it stop at 30 with a different number of turns?
ST2: by putting another (point to the spool)
INT: a different spool -- how would that work?
ST1: You take it out and put it back on
INT: no [INT interpreted the comment as being about the way to physically change the spool]
ST2: You put in a 3 and then turn it twice as many times
INT: So if you put in a 3 it would take --
ST2: Twice as many turns as if you used the 6.

...

INT: Okay, do you want to try it? Sure. Do you know it's going to work?
ST1: Yeah
ST2: Yeah, Well, it'll be pretty close. It'll be very close.
(INT changes the spool to 3-inch)
INT: So, what's going to happen now? What are you going to do?
ST1: We're going to pull it 6 times -- no, 10 times, and we'll get to 30.

The students generated the idea of using a different spool, indicating that they appreciated (a) that this variable could have different values, and (b) that a smaller value of one variable (the spool size) could be compensated by a larger value of another variable (the number of turns). In further questioning, these students indicated appreciation that the starting point (i.e., the intercepts of functions) could be varied, and discussed the effect of having different intercepts on the behavior of the function. We also asked what someone would have to be told to know (a) the ending position, (b), the starting position, (c) the spool size, and (d) the number of turns, and they clearly understood that for any of the four variables, the value could be inferred from values of the other three variables.

Evidence for students' understanding of functions also was obtained in the regular interviews. The sample that I discussed previously included evidence that all the student pairs understood that $f2(x) = 6x$ is monotonically greater than $f1(x) = 3x$ for $x > 0$. More specifically, all the students understood that the ratio of values of f2(x) to f1(x) would be two-to-one for at least one value of x, and all pairs except one recognized that the 2:1 ratio would hold for all $x > 0$. This constitutes implicit, situated understanding that would correspond to the symbolic operation of dividing 6x by 3x, and realizing that the resulting constant 2 is the ratio of values of the two functions. It is interesting that the students' understanding of the ratio of these functions appeared to be stronger than their understanding of the difference between them, which of course was not a constant. For the question, "How far ahead will the blue block be after four turns?" four pairs of students did not draw the correct inference initially, apparently not recognizing the way to arrive at a quantitative inference. Only one of the pairs did not appear to grasp the relation among the quantities after they saw the event. Two of the pairs of students did not give correct answers to the question, "Will the blue block ever be ahead by six?" Our general conclusion is that most students, including the seventh-graders, had good implicit understanding of relations between functions involving both ratios and distances, but the relation of proportionality was understood more easily in this situation. Indeed, several of the student pairs used ratios to obtain quantities that were then used to make inferences about differences. There is a widespread belief that reasoning in terms of proportions is much more difficult for adolescents than reasoning in terms of differences. Apparently that belief is too general. The relative difficulty of reasoning about proportionalities and differences apparently depends on the situation, and either kind of relation between quantities can be used, at least by 13- or 14-year-olds, if it is a salient feature of the situation so that they focus on it.

Situated reasoning. The second question that the research is focused on is the role of situational resources in reasoning about functional relations. The situations in which we can obtain evidence of implicit understanding provide a context that enables individuals to reason successfully. The question is: What is there about situations that does this?

The answer is not as obvious as it might seem. It is not sufficient to say simply that it is easier to reason about concrete objects than about more "abstract" entities such as properties and relations. In fact, what students reasoned about in the research tasks were quantitative properties and relations.

I believe that an adequate theory of this reasoning will recognize that properties and relations such as the positions of the blocks, the speeds of their individual motions, the distance between the blocks and changes in that distance can be directly perceived and reasoned about by individuals. This is consistent with Gibson's (1966) theoretical discussions of perception. The reasoning we asked students to do involved associating various of these features with numerical quantities. A reasonable conjecture is that the quantities and their relations could be observed as structural features of events in the situation, and that understanding their relations was supported by the evident causal relations between turning the handle, winding the string around the spool, and pulling the blocks along the track.

One obvious way in which the situation supported reasoning was in enabling students to run mental simulations of events, thereby inferring the state of the machine after some number of turns or inferring the number of turns it would take to reach a specified state. Some of these thought experiments included pointing to positions on one or both of the rulers; others involved simply saying numbers while looking at the machine, apparently focussing on the corresponding positions.

There is no reason to assume that students constructed fully elaborated mental images of events, complete with turning handles and blocks moving along the tracks. An alternative is that they simulated relations between the quantities involved in the events, including the number of turns and the distances that blocks moved, but in other cases, more complex quantitative features such as the distance between the blocks. This kind of simulative reasoning is interesting theoretically, because it is concerned with relatively complex features of situations, but it apparently depends on, or at least is facilitated by, being in the situation in order to do the reasoning. Therefore, it raises an important theoretical question, because in current theories, reasoning about complex properties such as quantitative relations generally is done with symbolic representations, but in this case, concrete objects and events seem to play an important role.

The students' reasoning focused on some quantitative properties and relations in preference to others. It is interesting that in this situation, ratios of quantities seem to have been more salient than differences, in contrast to many other situations that have been studied. We evidently need to analyze properties of the situations in which people reason to understand how they come to focus on some of the features, rather than others.

The situated character of this reasoning is clarified by some examples that have negative features. One example was given in the results, for the question, "How far ahead will the blue block be after four turns?" Our interpretation is that ST1's connection to the situation did not focus on the difference between the two blocks' positions, as the question indicated, but involved reasoning activities that were easier for ST1 to engage in, which involved the positions of the individual blocks rather than the difference between them. The "collaboration" between ST1 and ST2 was not entirely cooperative, but the interaction did lead to a shared focus on the difference between the positions when ST2 counted out that quantity.

Another example was provided by a pair of 11th-grade students. The questions was how to make both blocks arrive at 24 at the same time. ST1 proposed linking the axles, putting a 4-spool on each axle, and turning the handle six times. Then the interviewer asked for another solution.

INT: Ok, so everything's equal, right? Ok, how about if, let's say I have an unequal starting point, one starts ahead of the other.
ST2: Put one --
ST1: Put a --
ST2: -- at six, one at three, turn one eight, turn one four.
INT: Ok, can you figure out a specific answer. I'm sorry, maybe I missed that one. What was --
ST1: I think you just said --
ST2: You put one, put one, ok give it the spool of three.
INT: Ok
ST2: Put it at, um, where are you going to put it at?
ST1: eight
ST2: No, I screwed up.
ST1: No, you gotta take the spool --
ST2: Put one at eight, right?
ST1: Put it at eight.
ST2: And then turn that three times with the three spool (put the block on ST2's side at 8)
INT: Ok, we're trying to tie at 24.
ST2: Trying to tie at 24. All right, so put this one at four (put the block on ST1's side at 4)
ST1: Put a 6-spool on it
ST2: Turn it four times -- no, you need to turn it six times. And then you're tied at 24.
ST1: With a -- yeah.
ST2: With a --
ST1: But would we tie? Would we be there at the same time?
ST2: If we turn it slowly.
INT: You want to check that out?
ST1: But we don't have a 3-spool.
INT: Yeah you do, the small one's a three.
ST1: Oh, ok.
ST2: Now wait; I got six -- no, I got three. (ST1 place a 6-spool on an axle; ST2 place a 3-spool on an axle.)
ST1: You're more mechanically inclined than I am.
INT: You said start this one where? (adjust the block on ST2's side) At eight?
ST1: Eight, and this one (block on ST1's side) at four.

[ST1 and ST2 ask some questions about the reason for the study, the videotaping, and the mechanics of linking the axles]

ST1: So these shouldn't be hooked up then, should they?
INT: Oh, it's up to you. You're constructing the situations. So you wanted this one to be at eight, right? Ok, there's eight. (place ST2's block at 8) There's four (place ST1's block at 4) Now, you said --
ST2: Something's wrong.

INT: How many turns did you say?
ST2: Something's wrong.
INT: We're trying to tie --
ST2: What were we trying to get at again?
ST1: 24.
INT: We're trying to tie it right here (place a marker at 24)
ST2: It's not going to work.
INT: Why not? What's wrong?
ST2: I said, I screwed up; I said, um, you turn it, you have to turn it, you can't turn it an even, an even, I mean a whole number of times to get it to be eight.
INT: Oh, you don't have to turn a whole number of times.
ST2: Oh, just get it there?
INT: Yeah, well you have to tie at 24. that was the --
ST1: Do we have to turn it the same --
ST2: We both arrive at 24 at the same time?
INT: That's right, that's the only thing, that's your goal.

[some further discussion about the problem, with ST2 expressing some negative feelings about what's going on]

INT: What are you thinking?
ST2: I, if you put the six on, put it at the zero and turn four times, or put it at 12 and turn it two times
INT: Ok
ST2: Or put it at six and turn it three times
INT: Gotcha
ST2: Then you'll make it, but then you'd have to get the three, put it at, let's see if this (the 6-spool) takes -- four, three'll take -- eight. So put it halfway, so if we turn it equal number of times, so if I'm turning this (6-spool) four times and you turn this four times, three times four is 12, so put that at 12 (ST1's block), put this at zero (ST2's block), and turn 'em equal times.

[ST1 and ST2 then run a trial, confirming that these conditions work.]

Our interpretation is that ST2 was only weakly connected to the situation. He was acting quite bored. When the question was asked, getting both blocks to the position 24, ST2 generated two pairs of numbers with products of 24 (six times four and three times eight). This would have solved the problem if six and three were spool sizes and four and eight were numbers of turns. ST2, however, specified six and three as starting points. ST2 adjusted the solution by specifying different spool sizes. Eight became a starting point, with three both a spool size and a number of turns. The other block was placed at 4, and six was specified as both a spool size and a number

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of turns. The discussion, mainly by ST2, has the character of "playing with numbers," rather than reasoning about the quantities in the situation. Eventually, after a rather long excursion including a detour into a conversation about what the study was for, ST2 produced an analysis that was adequate to provide a solution. This happened after the machine had been set up with two spools and the handles were being turned. Therefore, it seems likely that the solution depended on being in the situation and observing the relations between quantities, although it is possible that it could have happened without the machine being physically present.

Language. The view presented earlier of language as a means for individuals to become similarly connected to situations is supported by several examples in the data.

One example is in the protocol given earlier that starts with ST1 saying, "I have something else: one turn is three inches, and three times four --." The question had been about how far ahead one block would be, and ST2 had focused on that quantity, so ST2 and INT were focused on the same feature of the situation. ST1, however, apparently did not focus on the difference between the positions, but rather on the positions of the separate blocks. The communicative intent of INT was not achieved, and it required the relatively extreme action by ST2 of counting the positions between the blocks before ST1 recognized what INT and ST2 meant.

An important issue about language is the generality of terms that are used to refer to features of the situation. For example, if a block moves a distance of six inches, that can be denoted in a specific way, say, "It went from nine to 15," or a more general way, "The distance it moved," or even, "The number of inches it moved." We believe that the ability to refer to quantitative features of situations in general terms may be an important factor in understanding mathematical principles meaningfully and for using mathematical knowledge in a broad range of situations.

We made a small start toward studying the generality of reference in this study by including the following question: "Now, imagine that blue is going to start behind red by some amount, but I won't tell you the amount, and blue has a bigger spool than red, but I won't tell you the spool sizes. The axles have to be linked. What can you say about when blue will catch up with red?" Many of the students were apparently mystified by this question, but there some of them worked through to quite a sensible response. For example, with one of the seventh-grade pairs, the exchange went as follows:

> ST1: Oh, well, we don't know. We have some missing information.
> INT: Ok, so that's the first thing. You can't, you can't--
> ST1: You can't figure it out
> ST2: Unless you have the right information.
> ST1: Unless we know, well, what --
> ST2: So if you tell us what spool --
> ST1: How behind and what spool.
> ST2: Yeah, if you told us which, what type of sppol this one had (point to red block) we could figure it out.
> INT: Ok, can you say how you would figure it out?
> ST2: More or less the same way as we've been doing, but also we'll need to know at what number the red one was, as another piece of information, otherwise we wouldn't get it completely correct.
> INT: Ok, so when you say you'd do it the same as you've been doing, can you say a little more, just so that I know what you've been doing?
> ST2: Yeah, um well, what we have been doing is the number on the spool, which is the thing going around. That we would add on to the number that it was, you know for each rotation we'd add that number on once, and then again, and we'd do the same thing for this one, with the red one as well, and when we'd write it down we'd get columns, once, and um, that's the way we've done it.

This pair of students had been solving problems mainly by constructing columns of numbers, comparing the positions after different numbers of turns or determining points of equality. In effect, ST2 specified that procedure in general terms.

A pair of 11th-grade students had a conversation with the interviewer that referred to functional relations in quite general terms, and also touched

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on some interesting epistemological issues.

INT: ... can you tell me, just knowing those facts, this spool's bigger and it's got a head start, can you tell me something about when blue will catch up with red?

ST2: Is it a nine head start? Or just any head start?

ST1: No, you don't know. A head start.

ST2: If this is six and this is three?

INT: No, no.

ST2: This is bigger, and this is --

INT: That's bigger

ST1: You could just give a theory --

ST2: How many turns is that how much, is that what you want?

ST1: You can just give a theory of how many times proportional to how big the spool is, but you can't really say exactly, you can't say this'll be --

INT: Ok, I was looking for this theoretical answer that you were talking about. That's what I was looking for.

ST2: Very good.

ST1: In theory you can talk about how, but you can't give an exact answer, you can only give theories on how, it will catch up, but you can't say exactly when because you don't know the sizes.

ST2: You could probably have an equation.

INT: Ok, so is the theory just that it's going to catch up, or can you say something more about it?

ST1: Well, it'll be proportional to the spool sizes of each, of the spool sizes, and the --

INT: Proportional?

ST1: Well, um, yeah. Well, um, yeah, well, it'll be related--

INT: That sounds really general.

ST1: Related's a less mathematical term. Um, yeah, it'll be, well, yeah, the time, the number of turns it takes for blue to catch up with red will depend on the size of spool

ST2: And the head start

ST1: and the head start.

INT: Ok, you told me these dependencies, but you haven't given me sort of a, a more concrete or specific expression of how those things --

ST2: What do you want, an equation?

INT: Well, you could give me that, or --

ST2: I don't know an equation.

INT: An equation implies that it has to be, you know, numbers and things.

ST2: You have to understand we're used to multiple choice tests and fill-ins. We're not used to --

INT: Ok.

ST2: We're not used to thinking.

ST1: We could write a theory down saying, do you want a theory? Would that be ok?

INT: Hey, whatever you want.

ST1: Create some great new scientific math theory or something. No, um, you could say that for every turn that the larger spool takes, the larger spool takes, it will therefore gain the amount that it is larger over the smaller spool.

INT: Ok, the proportion or the difference?

ST1: The difference between the two.

ST2: How about, how about the, um -- I don't know, this is a guess.
INT: And how do you tie that to, to the starting difference, to the starting lag?
ST1: Um, and depending on the head start, it'll take longer --
ST2: Go ahead, finish your sentence.
ST1: I'm not sure, I'm just, you know --
ST2: Maybe the larger the ratio of the spools, the larger the head start you're going to need?
INT: To --
ST2: To have 'em meet? Is that what we're, what are we looking for?
INT: The question was, yeah, when will blue catch up with red, yes. Yes, yeah, you're right -- when they will meet.
ST1: The larger the proportion between the spools, the shorter time it'll take for it to catch up with a given head start.

In addition to illustrating an ability, especially by ST1, to think about the system in general terms, the conversation shows that mutual understanding about the discussion itself can be important. The students and the interviewer had to negotiate the level of generality that was appropriate, and converge toward a set of terms that would be at a level that was agreeable to everyone as well as within the capabilities of the students.

Forms of Learning

One alternative to current instructional practice would involve a radical change in the role of students in the learning situation. Presently, students are individual recipients of mathematical knowledge. The alternative would make them participants in a social process of mathematical knowing.

This is a radical proposal, and one response to it is to say that it cannot work. One reason for thinking it cannot work is that people believe that nearly all students are incapable of mathematical reasoning until they have worked their way through the basic material of the curriculum. If that were true, then the proposal to make students the constructors of their mathematical knowledge would surely fail. But it may not be true, and if it is not, we need to find out what the capabilities of students for mathematical reasoning really are. The research that I have described here is an attempt to address the question of conceptual competence relevant to algebra. The preliminary findings that I have presented indicate that middle-school and high-school students have significant principled understanding of variables and functions. Findings like these encourage thinking about ways that the learning activity of children could be changed into a practice of mathematical knowing, including discussions of concepts and principles along with whatever practice it takes to acquire the skills of computational notation for reasoning as well as communication.

There is another question of course, beyond children's capabilities for mathematical reasoning. This other question is whether we can put together social environments in which students can engage in activities of a mathematical practice. Lampert's teaching (1986, 1988) demonstrates that that is possible, too. Her demonstration is very unusual, but it is not quite unique. Fawcett's (1938) high-school geometry class engaged in the practice of constructing geometry. Another contemporary example is Schoenfeld's (in press) teaching of mathematical problem solving for college undergraduates, where students engage in activities of mathematical reasoning and discovery. Lampert's demonstrations are particularly important because she works with elementary-school children. There is no need to wait until children have reached college -- or even middle school -- to engage them in a meaningful practice of mathematical reasoning.

I want to conclude by stating the proposal for a different form of learning in metaphorical terms. I will briefly characterize three models of teaching and learning mathematics.

In the first metaphor, mathematical teaching is like operating a pump. Mathematical knowledge is a sludgy substance that exists in the ground somewhere, and it's pumped out and refined into a version that we try to get students to consume. The refined versions are kept in tanks, and there's a distribution system that carries the stuff around to schools, something like the way that petroleum is transported to gas stations. Learning is like ingesting and digesting the stuff, so the teacher has the job of getting mathematical knowledge into the heads of the students. It takes a lot of effort to get this to happen. Learning only works if the stuff is supplied with a lot of pressure, so the teacher has to pump hard. This isn't easy, since the stuff is viscous and it tends to get stuck in the pipe. Besides that, teachers don't understand the physics and chemistry of the stuff very well, so they don't always know where to put the pipe into the tank to get some stuff that they can pump into the students. Students have to sit very still, and in exactly the right positions, to receive the stuff effectively, so they have to pay very close attention, not wiggle around, and do some special exercises so the digestive process works.

The pump-ingestion metaphor of teaching and learning is something none of us likes, and most of us worry that it's a truer picture of educational practice than we wish it was. In another metaphor, which seems better, learning involves building things, including machines that work and structures of materials that are well integrated, strong, and beautiful. Mathematical knowledge is a wonderful structure that has been built over the years. Teachers have been taken on tours of some parts of the structure, so they have some idea of what it looks like, but they haven't been engaged very much in the actual building of the structure. In schools, there are supplies of pieces that can be put together to form structures that are like parts of the real mathematical structure. Teachers give students these various pieces, along with instructions for building little machines and structures of the parts. Students construct things with the parts, and then show them to the teacher, who tells them whether the machines work the way they should, and whether the parts of the structures are connected to each other according to plans that the teacher has learned about.

The construction-workshop metaphor of teaching and learning has significant advantages compared to the pump-and-ingestion metaphor, but there is a third metaphor that I like even better. This metaphor focuses on mathematics as an activity and the teacher is an activities coordinator. Knowing mathematics includes abilities to participate in discussions of mathematical ideas, propose conjectures, evaluate these proposals by constructing arguments (some of which are put into the form of proofs), use the ideas and methods of those interactions to think about the world, construct interpretations and arguments about how the world should be changed, make things that are useful, and write papers about the theoretical as well as the practical aspects of the subject. These abilities aren't just carried around in the person's head; they are used in the person's participation in a community where the understanding and use of mathematical ideas are a continuing focus of activity. They also are used in the person's participation in situations where the focus is on some nonmathematical goal, but a mathematical idea or method can be used productively.

If we take seriously the idea that mathematics is an intellectual activity, engaged in with other people, we need to think about what it would mean to teach children to do that. If the goal of mathematics education were to give students abilities to participate meaningfully in activities of mathematical understanding and reasoning, including understanding and reasoning about situations in the world using mathematics as a resource, then their instructional environment should be one where that is what they do. In that case, the teacher's task is to lead and coordinate the activity of a group, so that their efforts are productive.

Mathematical structures and content do not disappear when we adopt the view of mathematical knowledge as participation in activity. Structures of concepts and computation are what mathematical activity is for. That is, what the active mathematicians in the third metaphor are doing is building structures of concepts, discovering ways to use those structures and concepts, and developing methods for both building and using the structures. There is even a certain amount of the gucky stuff of routine procedures involved, although these can be functional in the context of sense-making and meaning construction, rather than being the main focus of the enterprise.

The three kinds of teaching and learning that I've caricatured in the metaphors are not just different ways of teaching and learning the same thing. These different forms of teaching and learning lead to profoundly different forms of knowing that students will be capable of. If they are told to sit still and receive mathematical knowledge, the knowledge they acquire is likely to be like Figure 3, where the symbols of mathematics only refer to themselves. They also are likely to construct an epistemology in which mathematics is received knowledge, warranted only by the authority of experts and textbooks. A teaching and learning situation where students construct their knowledge has a better chance of resulting in significant capabilities of knowing and reasoning; however, the blueprints and judgments of adequacy are still supplied to the students by others. At this point, it seems reasonable to conjecture that to achieve the goals of meaningful learning and productive epistemological outcomes of mathematics education, learning environments like those developed by Lampert, Fawcett, and Schoenfeld will have to become commonplace, rather than exceptional.

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